

QUANTUM COHOMOLOGY OF THE COTANGENT BUNDLE OF A FLAG VARIETY AS A YANGIAN BETHE ALGEBRA

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ABSTRACT. We interpret the equivariant cohomology algebra $H_{GL_n \times \mathbb{C}^*}^*(T^*\mathcal{F}_\lambda; \mathbb{C})$ of the cotangent bundle of a partial flag variety \mathcal{F}_λ parametrizing chains of subspaces $0 = F_0 \subset F_1 \subset \dots \subset F_N = \mathbb{C}^n$, $\dim F_i/F_{i-1} = \lambda_i$, as the Yangian Bethe algebra $\mathcal{B}^\infty(\frac{1}{D}\mathcal{V}_\lambda^-)$ of the \mathfrak{gl}_N -weight subspace $\frac{1}{D}\mathcal{V}_\lambda^-$ of a $Y(\mathfrak{gl}_N)$ -module $\frac{1}{D}\mathcal{V}^-$. Under this identification the dynamical connection of [TV1] turns into the quantum connection of [BMO] and [MO]. As a result of this identification we describe the algebra of quantum multiplication on $H_{GL_n \times \mathbb{C}^*}^*(T^*\mathcal{F}_\lambda; \mathbb{C})$ as the algebra of functions on fibers of a discrete Wronski map. In particular this gives generators and relations of that algebra. This identification also gives us hypergeometric solutions of the associated quantum differential equation. That fact manifests the Landau–Ginzburg mirror symmetry for the cotangent bundle of the flag variety.

1. INTRODUCTION

A Bethe algebra of a quantum integrable model is a commutative algebra of linear operators (Hamiltonians) acting on the space of states of the model. An interesting problem is to describe the Bethe algebra as the algebra of functions on a suitable scheme. Such a description can be considered as an instance of the geometric Langlands correspondence, see for example [MTV2, MTV3]. The \mathfrak{gl}_N XXX model is a quantum integrable model. The Bethe algebra \mathcal{B}^q of the XXX model is a commutative subalgebra of the Yangian $Y(\mathfrak{gl}_N)$. The algebra \mathcal{B}^q depends on the parameters $q = (q_1, \dots, q_N) \in \mathbb{C}^N$. Having a $Y(\mathfrak{gl}_N)$ -module M , one obtains the commutative subalgebra $\mathcal{B}^q(M) \subset \text{End}(M)$ as the image of \mathcal{B}^q . The geometric interpretation of the algebra $\mathcal{B}^q(M)$ as the algebra of functions on a scheme leads to interesting objects, see for example, [MTV4].

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In this paper we consider an extension of the Yangian $Y(\mathfrak{gl}_N)$, denoted $\tilde{Y}(\mathfrak{gl}_N)$, which is a subalgebra of $Y(\mathfrak{gl}_N) \otimes \mathbb{C}[h]$, and we work with the corresponding Bethe subalgebra $\tilde{\mathcal{B}}^q \subset \tilde{Y}(\mathfrak{gl}_N)$.

One of the most interesting Yangian modules is the vector space $\mathcal{V} = V \otimes \mathbb{C}[z_1, \dots, z_n, h]$ of V -valued polynomials in z_1, \dots, z_n , where $V = (\mathbb{C}^N)^{\otimes n}$ is the tensor power of the standard vector representation of \mathfrak{gl}_N . We introduce on \mathcal{V} two Yangian actions, called ϕ^\pm , and two actions of the symmetric group S_n , called S_n^\pm -actions. The Yangian action ϕ^+ commutes with the S_n^+ -action. Hence, the subspace $\mathcal{V}^+ \subset \mathcal{V}$ of S_n^+ -invariants is a Yangian module. Similarly, the Yangian action ϕ^- commutes with the S_n^- -action. Hence, the subspace $\mathcal{V}^- \subset \mathcal{V}$ of S_n^- -skew-invariants is a Yangian module. The Yangian module structure on \mathcal{V}^- induces a Yangian module structure on $\frac{1}{D}\mathcal{V}^- = \{\frac{1}{D}f \mid f \in \mathcal{V}^-\}$, where $D = \prod_{1 \leq i < j \leq n} (z_i - z_j + h)$. The Bethe algebra $\tilde{\mathcal{B}}^q$ preserves the \mathfrak{gl}_N -weight decompositions $\mathcal{V}^+ = \bigoplus_{\lambda} \mathcal{V}_{\lambda}^+$ and $\frac{1}{D}\mathcal{V}^- = \bigoplus_{\lambda} \frac{1}{D}\mathcal{V}_{\lambda}^-$, $\lambda = (\lambda_1, \dots, \lambda_N) \in \mathbb{Z}_{\geq 0}^N$, $|\lambda| = n$.

In this paper we study the limit of the algebras $\tilde{\mathcal{B}}^q(\mathcal{V}_{\lambda}^+)$, $\tilde{\mathcal{B}}^q(\frac{1}{D}\mathcal{V}_{\lambda}^-)$ as q tends to infinity so that $q_{i+1}/q_i \rightarrow 0$ for all $i = 1, \dots, N-1$, and $q_N = 1$. We show that in this limit each of the Bethe algebras $\tilde{\mathcal{B}}^\infty(\mathcal{V}_{\lambda}^+)$, $\tilde{\mathcal{B}}^\infty(\frac{1}{D}\mathcal{V}_{\lambda}^-)$ can be identified with the equivariant cohomology algebra $H_{\lambda} := H_{GL_n \times \mathbb{C}^*}^*(T^*\mathcal{F}_{\lambda}; \mathbb{C})$ of the cotangent bundle of the partial flag variety \mathcal{F}_{λ} parametrizing chains of subspaces

$$0 = F_0 \subset F_1 \subset \dots \subset F_N = \mathbb{C}^n, \quad \dim F_i/F_{i-1} = \lambda_i.$$

More precisely, we construct an algebra isomorphism $\mu_{\lambda}^+ : H_{\lambda} \rightarrow \tilde{\mathcal{B}}^\infty(\mathcal{V}_{\lambda}^+)$ and a vector space isomorphism $\nu_{\lambda}^+ : H_{\lambda} \rightarrow \mathcal{V}_{\lambda}^+$, which identify the $\tilde{\mathcal{B}}^\infty(\mathcal{V}_{\lambda}^+)$ -module \mathcal{V}_{λ}^+ with the action of the algebra H_{λ} on itself by multiplication operators, see Theorem 5.4. Similarly, we construct an algebra isomorphism $\mu_{\lambda}^- : H_{\lambda} \rightarrow \tilde{\mathcal{B}}^\infty(\frac{1}{D}\mathcal{V}_{\lambda}^-)$ and a vector space isomorphism $\nu_{\lambda}^- : H_{\lambda} \rightarrow \frac{1}{D}\mathcal{V}_{\lambda}^-$, which identify the $\tilde{\mathcal{B}}^\infty(\frac{1}{D}\mathcal{V}_{\lambda}^-)$ -module $\frac{1}{D}\mathcal{V}_{\lambda}^-$ with the action of the algebra H_{λ} on itself by multiplication operators, see Theorem 5.6.

In Section 6, using the discrete Wronski determinant $W^q(u)$, see formula (6.1), we introduce an algebra \mathcal{H}_{λ}^q and construct vector space isomorphisms $\mu_{\lambda}^{q+} : \mathcal{H}_{\lambda}^q \rightarrow \mathcal{V}_{\lambda}^+$, $\mu_{\lambda}^{q-} : \mathcal{H}_{\lambda}^q \rightarrow \frac{1}{D}\mathcal{V}_{\lambda}^-$ and algebra isomorphisms $\nu_{\lambda}^{q+} : \mathcal{H}_{\lambda}^q \rightarrow \tilde{\mathcal{B}}^q(\mathcal{V}_{\lambda}^+)$, $\nu_{\lambda}^{q-} : \mathcal{H}_{\lambda}^q \rightarrow \tilde{\mathcal{B}}^q(\frac{1}{D}\mathcal{V}_{\lambda}^-)$. The isomorphisms μ_{λ}^{q+} and ν_{λ}^{q+} identify the $\tilde{\mathcal{B}}^q(\mathcal{V}_{\lambda}^+)$ -module \mathcal{V}_{λ}^+ and the regular representation of \mathcal{H}_{λ}^q , see Theorem 6.3. Similarly, the isomorphisms μ_{λ}^{q-} and ν_{λ}^{q-} identify the $\tilde{\mathcal{B}}^q(\frac{1}{D}\mathcal{V}_{\lambda}^-)$ -module $\frac{1}{D}\mathcal{V}_{\lambda}^-$ and the regular representation of \mathcal{H}_{λ}^q , see Theorem 6.5. In particular, the algebras $\tilde{\mathcal{B}}^q(\mathcal{V}_{\lambda}^+)$ and $\tilde{\mathcal{B}}^q(\frac{1}{D}\mathcal{V}_{\lambda}^-)$ are isomorphic, and the $\tilde{\mathcal{B}}^q(\mathcal{V}_{\lambda}^+)$ -module \mathcal{V}_{λ}^+ is isomorphic to the $\tilde{\mathcal{B}}^q(\frac{1}{D}\mathcal{V}_{\lambda}^-)$ -module $\frac{1}{D}\mathcal{V}_{\lambda}^-$.

Under the isomorphisms $\nu_{\lambda}^+ : H_{\lambda} \rightarrow \mathcal{V}^+$ and $\nu_{\lambda}^- : H_{\lambda} \rightarrow \frac{1}{D}\mathcal{V}^-$ the subalgebras $\tilde{\mathcal{B}}^q(\mathcal{V}_{\lambda}^+)$ and $\tilde{\mathcal{B}}^q(\frac{1}{D}\mathcal{V}_{\lambda}^-)$ induce respectively commutative subalgebras $\tilde{\mathcal{B}}_{\lambda}^{q+}$ and $\tilde{\mathcal{B}}_{\lambda}^{q-}$ of $\text{End}(H_{\lambda})$. Theorem 7.10 says that the Bethe algebra $\tilde{\mathcal{B}}_{\lambda}^{q-}$ is the algebra $QH_{GL_n \times \mathbb{C}^*}(T^*\mathcal{F}_{\lambda}; \mathbb{C})$ of quantum multiplication on H_{λ} described in [BMO, MO] under the identification of notation (7.4). The proof of Theorem 7.10 is based on [RTV], where it is shown that the Yangian structure on the equivariant cohomology defined in [MO] coincides with the Yangian structure ρ^- defined

in Section 5.4. In Section 8 we give a direct proof of the special case of Theorem 7.10 when $N = n$, $\lambda = (1, \dots, 1)$. The proof is based on formulae from [BMO].

In the first version of this paper, see [arXiv:1204.5138v1](#), the statement of Theorem 7.10 was formulated as a conjecture. The later paper [MO] contained, in particular, a description of the quantum cohomology algebra $QH_{GL_n \times \mathbb{C}^*}(T^*\mathcal{F}_\lambda; \mathbb{C})$. That allowed us to relate the quantum cohomology algebra and the Bethe algebra $\tilde{\mathcal{B}}_\lambda^{q-}$.

Corollary 7.11 gives a description of the quantum cohomology algebra $QH_{GL_n \times \mathbb{C}^*}(T^*\mathcal{F}_\lambda; \mathbb{C})$ by generators and relations. Theorem 7.13 shows that in the limit $h \rightarrow \infty$ such that $h^{\lambda_i + \lambda_{i+1}} q_{i+1}/q_i$ are fixed for all $i = 1, \dots, N-1$, the description of $QH_{GL_n \times \mathbb{C}^*}(T^*\mathcal{F}_\lambda; \mathbb{C})$ turns into the known description of the quantum cohomology algebra $QH_{GL_n}(\mathcal{F}_\lambda; \mathbb{C})$ of the partial flag variety \mathcal{F}_λ , see [AS].

In more general terms, our identification of the quantum multiplication on H_λ and the Yangian Bethe algebra $\tilde{\mathcal{B}}^{q-}$ indicates a relation between quantum integrable chain models and quantum cohomology algebras, see also the paper [MO] (which appeared later than the first version of this paper). That relation was discussed in physics literature in recent works by Nekrasov and Shatashvili, see for example [NS].

Note that a relationship between quantum integrable models and quantum cohomology/WZNW fusion rings has been shown also in [K, KoS], see references therein.

The identification of the quantum multiplication on H_λ and the Yangian Bethe algebra $\tilde{\mathcal{B}}^{q-}$ gives a relation between interesting objects in both theories. For example, the eigenvectors of the Bethe algebra is the main object of the XXX model and finding eigenvectors is the subject of the well developed Bethe ansatz theory. Under the above identification the eigenvectors correspond to idempotents of the quantum multiplication and the idempotents are important in the theory of Frobenius manifolds, see [D]. According to the above identification, the idempotents of the quantum cohomology algebra can be determined by the XXX Bethe ansatz method, see an example in Section 8.4.

The above identification brings another interesting relation. Namely, Theorem 7.10 shows that the isomorphism ν_λ^- identifies the trigonometric dynamical connection of [TV1] with the quantum connection of [BMO, MO]. It is known that the flat sections of the trigonometric dynamical connection are given by multidimensional hypergeometric integrals, see [TV1, MV, SV], cf. [TV2]. These hypergeometric integrals provide flat sections of the quantum connection of [MO, BMO], see [TV3]. This presentation of flat sections of the quantum connection as hypergeometric integrals is in the spirit of the mirror symmetry, see Candelas et al. [COGP], Givental [G1, G2], and [BCK, BCKS, GKLO, I, JK], see also an example in Section 8.4.

The results of this paper are parallel to the results of the paper [RSTV], where we consider the Bethe subalgebra of $U(\mathfrak{gl}_N[t])$ instead of the Bethe algebra of $\tilde{Y}(\mathfrak{gl}_N)$.

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2. SPACES \mathcal{V}^+ AND $\frac{1}{D}\mathcal{V}^-$

2.1. Lie algebra \mathfrak{gl}_N . Let $e_{i,j}$, $i, j = 1, \dots, N$, be the standard generators of the Lie algebra \mathfrak{gl}_N satisfying the relations $[e_{i,j}, e_{k,l}] = \delta_{j,k}e_{i,l} - \delta_{i,l}e_{k,j}$. We denote by $\mathfrak{h} \subset \mathfrak{gl}_N$ the subalgebra generated by $e_{i,i}$, $i = 1, \dots, N$. For a Lie algebra \mathfrak{g} , we denote by $U(\mathfrak{g})$ the universal enveloping algebra of \mathfrak{g} .

A vector v of a \mathfrak{gl}_N -module M has weight $\lambda = (\lambda_1, \dots, \lambda_N) \in \mathbb{C}^N$ if $e_{i,i}v = \lambda_i v$ for $i = 1, \dots, N$. We denote by $M_\lambda \subset M$ the weight subspace of weight λ .

Let \mathbb{C}^N be the standard vector representation of \mathfrak{gl}_N with basis v_1, \dots, v_N such that $e_{i,j}v_k = \delta_{j,k}v_i$ for all i, j, k . A tensor power $V = (\mathbb{C}^N)^{\otimes n}$ of the vector representation has a basis given by the vectors $v_{i_1} \otimes \dots \otimes v_{i_n}$, where $i_j \in \{1, \dots, N\}$. Every such sequence (i_1, \dots, i_n) defines a decomposition $I = (I_1, \dots, I_N)$ of $\{1, \dots, n\}$ into disjoint subsets I_1, \dots, I_N , where $I_j = \{k \mid i_k = j\}$. We denote the basis vector $v_{i_1} \otimes \dots \otimes v_{i_n}$ by v_I .

Let

$$V = \bigoplus_{\lambda \in \mathbb{Z}_{\geq 0}^N, |\lambda|=n} V_\lambda$$

be the weight decomposition. Denote \mathcal{I}_λ the set of all indices I with $|I_j| = \lambda_j$, $j = 1, \dots, N$. The vectors $\{v_I, I \in \mathcal{I}_\lambda\}$ form a basis of V_λ . The dimension of V_λ equals the multinomial coefficient $d_\lambda := \frac{n!}{\lambda_1! \dots \lambda_N!}$.

Let \mathcal{S} be the bilinear form on V such that the basis $\{v_I\}$ is orthonormal. We call \mathcal{S} the Shapovalov form.

2.2. S_n -actions. Define an action of the symmetric group S_n on \mathcal{I}_λ . Let $I = (I_1, \dots, I_N) \in \mathcal{I}_\lambda$, where $I_j = \{i_1, \dots, i_{\lambda_j}\} \subset \{1, \dots, n\}$. For $\sigma \in S_n$, define $\sigma(I) = (\sigma(I_1), \dots, \sigma(I_N))$, where $\sigma(I_j) = \{\sigma(i_1), \dots, \sigma(i_{\lambda_j})\}$.

Let $P^{(i,j)}$ be the permutation of the i -th and j -th factors of $V = (\mathbb{C}^N)^{\otimes n}$. Define two S_n -actions on V -valued functions of z_1, \dots, z_n, h , called S_n^\pm -actions. For the S_n^+ -action, the i -th elementary transposition $s_i \in S_n$ acts by the formula:

$$(2.1) \quad s_i : f(z_1, \dots, z_n, h) \mapsto \frac{(z_i - z_{i+1})P^{(i,i+1)} - h}{z_i - z_{i+1}} f(z_1, \dots, z_{i+1}, z_i, \dots, z_n, h) + \\ + \frac{h}{z_i - z_{i+1}} f(z_1, \dots, z_i, z_{i+1}, \dots, z_n, h).$$

For the S_n^- -action, the i -th elementary transposition $s_i \in S_n$ acts by the formula:

$$(2.2) \quad s_i : f(z_1, \dots, z_n, h) \mapsto \frac{(z_i - z_{i+1})P^{(i,i+1)} + h}{z_i - z_{i+1}} f(z_1, \dots, z_{i+1}, z_i, \dots, z_n, h) - \\ - \frac{h}{z_i - z_{i+1}} f(z_1, \dots, z_i, z_{i+1}, \dots, z_n, h).$$

Lemma 2.1. *The S_n^\pm -actions (2.1) and (2.2) are well-defined.* □

Define operators $\tilde{s}_1^\pm, \dots, \tilde{s}_{n-1}^\pm$ acting on V -valued functions of z_1, \dots, z_n, h by

$$(2.3) \quad \tilde{s}_i^\pm f(z_1, \dots, z_n, h) = \frac{(z_i - z_{i+1}) P^{(i, i+1)} \mp h}{z_i - z_{i+1} \mp h} f(z_1, \dots, z_{i+1}, z_i, \dots, z_n, h).$$

Lemma 2.2. *The assignment $s_i \mapsto \tilde{s}_i^+$, $i = 1, \dots, n-1$, defines an action of S_n .* \square

Lemma 2.3. *The assignment $s_i \mapsto \tilde{s}_i^-$, $i = 1, \dots, n-1$, defines an action of S_n .* \square

Lemma 2.4. *The function $f(z_1, \dots, z_n, h)$ is invariant with respect to the S_n^+ -action (resp. S_n^- -action) if and only if $f = \tilde{s}_i^+ f$ (resp. $f = \tilde{s}_i^- f$) for every $i = 1, \dots, n-1$.* \square

Define operators $\hat{s}_1^\pm, \dots, \hat{s}_{n-1}^\pm$ acting on functions of z_1, \dots, z_n, h by

$$(2.4) \quad \begin{aligned} \hat{s}_i^\pm f(z_1, \dots, z_n, h) &= \frac{z_i - z_{i+1} \pm h}{z_i - z_{i+1}} f(z_1, \dots, z_{i+1}, z_i, \dots, z_n, h) \mp \\ &\mp \frac{h}{z_i - z_{i+1}} f(z_1, \dots, z_i, z_{i+1}, \dots, z_n, h) \end{aligned}$$

Lemma 2.5. *The assignment $s_i \mapsto \hat{s}_i^+$, $i = 1, \dots, n-1$, defines an action of S_n .* \square

Lemma 2.6. *The assignment $s_i \mapsto \hat{s}_i^-$, $i = 1, \dots, n-1$, defines an action of S_n .* \square

Let $f(z_1, \dots, z_n, h)$ be a V -valued function with coordinates $\{f_I(z_1, \dots, z_n, h)\}$:

$$f(z_1, \dots, z_n, h) = \sum_I f_I(z_1, \dots, z_n, h) v_I.$$

Lemma 2.7. *The function $f(z_1, \dots, z_n, h)$ is invariant with respect to the S_n^+ -action (resp. S_n^- -action) if and only if $f_{\sigma(I)} = \hat{\sigma}^+ f_I$ (resp. $f_{\sigma(I)} = \hat{\sigma}^- f_I$) for any $I \in \mathcal{I}_\lambda$ and any $\sigma \in S_n$.* \square

Lemma 2.8. *The function $f(z_1, \dots, z_n, h)$ is skew-invariant with respect to the S_n^+ -action (resp. S_n^- -action) if and only if $f_{\sigma(I)} = (-1)^\sigma \hat{\sigma}^- f_I$ (resp. $f_{\sigma(I)} = (-1)^\sigma \hat{\sigma}^+ f_I$) for any $I \in \mathcal{I}_\lambda$ and any $\sigma \in S_n$.* \square

Let

$$D = \prod_{1 \leq i < j \leq n} (z_i - z_j + h), \quad \check{D} = \prod_{1 \leq i < j \leq n} (z_j - z_i + h).$$

Lemma 2.9. *The function f is skew-invariant with respect to the S_n^+ -action (resp. S_n^- -action) if and only if the function $\frac{1}{D}f$ (resp. $\frac{1}{\check{D}}f$) is invariant with respect to the S_n^+ -action (resp. S_n^- -action).* \square

2.3. Spaces \mathcal{V}^+ , $\frac{1}{D}\mathcal{V}^=$ and $\frac{1}{D}\mathcal{V}^-$. Let $\mathcal{V}^+ = (V \otimes \mathbb{C}[z_1, \dots, z_n, h])^{S_n^+}$ be the space of S_n^+ -invariant V -valued polynomials. Let $\frac{1}{D}\mathcal{V}^=$ be the space of S_n^+ -invariant V -valued functions of the form $\frac{1}{D}f$, $f \in V \otimes \mathbb{C}[z_1, \dots, z_n, h]$. Let $\frac{1}{D}\mathcal{V}^-$ be the space of S_n^- -invariant V -valued functions of the form $\frac{1}{D}f$, $f \in V \otimes \mathbb{C}[z_1, \dots, z_n, h]$. All \mathcal{V}^+ , $\frac{1}{D}\mathcal{V}^=$ and $\frac{1}{D}\mathcal{V}^-$ are $\mathbb{C}[z_1, \dots, z_n]^{S_n} \otimes \mathbb{C}[h]$ -modules.

The S^\pm -actions on $V \otimes \mathbb{C}[z_1, \dots, z_n, h]$ commute with the \mathfrak{gl}_N -action. Hence \mathcal{V}^+ , $\frac{1}{D}\mathcal{V}^=$ and $\frac{1}{D}\mathcal{V}^-$ are \mathfrak{gl}_N -modules. Consider the \mathfrak{gl}_N -weight decompositions

$$\mathcal{V}^+ = \bigoplus_{\substack{\lambda \in \mathbb{Z}_{\geq 0}^N \\ |\lambda|=n}} \mathcal{V}_\lambda^+, \quad \frac{1}{D}\mathcal{V}^= = \bigoplus_{\substack{\lambda \in \mathbb{Z}_{\geq 0}^N \\ |\lambda|=n}} \frac{1}{D}\mathcal{V}_\lambda^=, \quad \frac{1}{D}\mathcal{V}^- = \bigoplus_{\substack{\lambda \in \mathbb{Z}_{\geq 0}^N \\ |\lambda|=n}} \frac{1}{D}\mathcal{V}_\lambda^-.$$

Define a partial ordering on subsets of $\{1, \dots, n\}$ of the same cardinality. Say $A \leq B$ if $A = \{a_1 < \dots < a_i\}$, $B = \{b_1 < \dots < b_i\}$, and $a_j \leq b_j$ for all $j = 1, \dots, i$.

Fix $\lambda = (\lambda_1, \dots, \lambda_N) \in \mathbb{Z}_{\geq 0}^N$, $|\lambda| = n$. For $I, J \in \mathcal{I}_\lambda$, say $I < J$ if $I_k = J_k$ for $k = 1, \dots, l-1$ with some l , and $I_l < J_l$. Define $I^{\min}, I^{\max} \in \mathcal{I}_\lambda$ by

$$(2.5) \quad I^{\min} = (\{1, \dots, \lambda_1\}, \{\lambda_1 + 1, \dots, \lambda_1 + \lambda_2\}, \dots, \{n - \lambda_N + 1, \dots, n\}), \\ I^{\max} = (\{n - \lambda_1 + 1, \dots, n\}, \{n - \lambda_1 - \lambda_2 + 1, \dots, n - \lambda_1\}, \dots, \{1, \dots, \lambda_N\}).$$

Clearly, $I^{\min} \leq I \leq I^{\max}$ for any $I \in \mathcal{I}_\lambda$.

Given $I \in \mathcal{I}_\lambda$, denote

$$(2.6) \quad Q(z_I) = \prod_{1 \leq a < b \leq N} \prod_{i \in I_a} \prod_{j \in I_b} (z_i - z_j + h).$$

For any function $f(z_1, \dots, z_n, h)$, set $\check{f}(z_1, \dots, z_n, h) = f(z_n, \dots, z_1, h)$. Let

$$(2.7) \quad Q_\lambda(z_1, \dots, z_n, h) = Q(z_{I^{\min}}),$$

so that $\check{Q}_\lambda(z_1, \dots, z_n, h) = Q(z_{I^{\max}})$.

Define

$$(2.8) \quad \vartheta_\lambda^+ : \mathbb{C}[z_1, \dots, z_n]^{S_{\lambda_1} \times \dots \times S_{\lambda_N}} \otimes \mathbb{C}[h] \rightarrow \mathcal{V}_\lambda^+, \\ \vartheta_\lambda^+(f) = \frac{1}{\lambda_1! \dots \lambda_N!} \sum_{\sigma \in S_n} \hat{\sigma}^{+\check{f}} v_{\sigma(I^{\max})},$$

$$(2.9) \quad \vartheta_\lambda^- : \mathbb{C}[z_1, \dots, z_n]^{S_{\lambda_1} \times \dots \times S_{\lambda_N}} \otimes \mathbb{C}[h] \rightarrow \frac{1}{D}\mathcal{V}_\lambda^=, \\ \vartheta_\lambda^-(f) = \frac{1}{\lambda_1! \dots \lambda_N!} \sum_{\sigma \in S_n} \hat{\sigma}^+(\frac{1}{Q_\lambda}\check{f}) v_{\sigma(I^{\max})},$$

and

$$(2.10) \quad \vartheta_\lambda^- : \mathbb{C}[z_1, \dots, z_n]^{S_{\lambda_1} \times \dots \times S_{\lambda_N}} \otimes \mathbb{C}[h] \rightarrow \frac{1}{D}\mathcal{V}_\lambda^-, \\ \vartheta_\lambda^-(f) = \frac{1}{\lambda_1! \dots \lambda_N!} \sum_{\sigma \in S_n} \hat{\sigma}^-(\frac{1}{Q_\lambda}f) v_{\sigma(I^{\min})}.$$

Lemma 2.10. *The maps ϑ_λ^+ , $\vartheta_\lambda^=$, and ϑ_λ^- are isomorphisms of the $\mathbb{C}[z_1, \dots, z_n]^{S_n} \otimes \mathbb{C}[h]$ -module $\mathbb{C}[z_1, \dots, z_n]^{S_{\lambda_1} \times \dots \times S_{\lambda_N}} \otimes \mathbb{C}[h]$ with the $\mathbb{C}[z_1, \dots, z_n]^{S_n} \otimes \mathbb{C}[h]$ -modules \mathcal{V}_λ^+ , $\frac{1}{D}\mathcal{V}_\lambda^=$ and $\frac{1}{D}\mathcal{V}_\lambda^-$, respectively.*

Proof. The operators $\hat{\sigma}^+$, $\sigma \in S_n$, preserve $\mathbb{C}[z_1, \dots, z_n, h]$. Thus $\vartheta_{\lambda}^+(f) = \sum_I f_I v_I$, and $f_I \in \mathbb{C}[z_1, \dots, z_n, h]$ for any I . Moreover, $f_{\sigma(I)} = \hat{\sigma}^+ f_I$ for any $\sigma \in S_n$, and $f_{I^{\max}} = \check{f}$. Hence, the map ϑ_{λ}^+ is well-defined by Lemma 2.7. If $g \in \mathcal{V}_{\lambda}^+$, $g = \sum_I g_I v_I$, then $g = \vartheta_{\lambda}^+(\check{g}_{I^{\max}})$ by Lemma 2.7, so ϑ_{λ}^+ is an isomorphism.

The proof for the cases of ϑ_{λ}^- and $\vartheta_{\lambda}^{\pm}$ are similar because the operators $\hat{\sigma}^+$, $\sigma \in S_n$, preserve the localized algebra $\mathbb{C}[z_1, \dots, z_n, h, \check{D}^{-1}]$, and the operators $\hat{\sigma}^-$, $\sigma \in S_n$, preserve the localized algebra $\mathbb{C}[z_1, \dots, z_n, h, D^{-1}]$. \square

It is known in Schubert calculus that $\mathbb{C}[z_1, \dots, z_n]^{S_{\lambda_1} \times \dots \times S_{\lambda_N}}$ is a free $\mathbb{C}[z_1, \dots, z_n]$ -module of rank d_{λ} . This yields that for any λ , the subspaces \mathcal{V}_{λ}^+ , $\frac{1}{D}\mathcal{V}_{\lambda}^-$ and $\frac{1}{D}\mathcal{V}_{\lambda}^{\pm}$ are free $\mathbb{C}[z_1, \dots, z_n]^{S_n} \otimes \mathbb{C}[h]$ -modules of rank d_{λ} .

The Shapovalov form \mathcal{S} on V induces a pairing

$$(2.11) \quad (V \otimes \mathbb{C}[z_1, \dots, z_n, h]) \otimes (V \otimes \mathbb{C}[z_1, \dots, z_n, h]) \rightarrow \mathbb{C}[z_1, \dots, z_n, h]$$

denoted by the same letter.

Lemma 2.11. *The pairing (2.11) induces a pairing*

$$(2.12) \quad \mathcal{V}^+ \otimes \frac{1}{D}\mathcal{V}^- \rightarrow \mathbb{C}[z_1, \dots, z_n]^{S_n} \otimes \mathbb{C}[h].$$

Proof. For any $f \in \mathcal{V}^+$ and $g \in \frac{1}{D}\mathcal{V}^-$, the scalar function $\mathcal{S}(f, g)$ is a symmetric function of z_1, \dots, z_n with possible poles only at the hyperplanes $z_i - z_j + h = 0$ for $1 \leq i < j \leq n$. Since this arrangement of hyperplanes is not invariant under the permutations of z_1, \dots, z_n , the poles are absent. \square

Lemma 2.12. *For any λ , the pairing $\mathcal{V}_{\lambda}^+ \otimes \frac{1}{D}\mathcal{V}_{\lambda}^- \rightarrow \mathbb{C}[z_1, \dots, z_n]^{S_n} \otimes \mathbb{C}[h]$, induced by pairing (2.12), is surjective.*

Lemma 2.12 is proved in Section 4.1

The pairing (2.11) also induces a pairing

$$(2.13) \quad \frac{1}{D}\mathcal{V}^{\pm} \otimes \frac{1}{D}\mathcal{V}^{\mp} \rightarrow Z^{-1} \mathbb{C}[z_1, \dots, z_n]^{S_n} \otimes \mathbb{C}[h],$$

where

$$(2.14) \quad Z = D\check{D} = \prod_{\substack{i,j=1 \\ i \neq j}}^n (z_i - z_j + h).$$

The pairing (2.12) and (2.13) will also be denoted by \mathcal{S} .

Lemma 2.13. *The pairings (2.12) and (2.13) are nondegenerate. That is, if $\mathcal{S}(f, g) = 0$ for a given $f \in \mathcal{V}^+$ and every $g \in \frac{1}{D}\mathcal{V}^-$, then $f = 0$. And similarly, for all other possible cases.* \square

2.4. Vectors ξ_I^{\pm} . Fix $\lambda = (\lambda_1, \dots, \lambda_N) \in \mathbb{Z}_{\geq 0}$, $|\lambda| = n$. Given $I \in \mathcal{I}_{\lambda}$, denote

$$(2.15) \quad R(z_I) = \prod_{1 \leq a < b \leq N} \prod_{i \in I_a} \prod_{j \in I_b} (z_i - z_j).$$

Proposition 2.14. *There exist unique elements $\{\xi_I^+ \in V \otimes \mathbb{C}[z_1, \dots, z_n, h, \check{D}^{-1}] \mid I \in \mathcal{I}_\lambda\}$ such that $\xi_{I^{\min}}^+ = v_{I^{\min}}$ and*

$$(2.16) \quad \xi_{s_i(I)}^+ = \tilde{s}_i^+ \xi_I^+$$

for every $I \in \mathcal{I}_\lambda$ and $i = 1, \dots, n-1$. Moreover,

$$(2.17) \quad \xi_I^+ = \sum_{J \leq I} X_{I,J}^+ v_J,$$

where $X_{I,J}^+ \in \mathbb{C}[z_1, \dots, z_n, h, \check{D}^{-1}]$, $X_{I,I}^+ \neq 0$, and

$$(2.18) \quad X_{I^{\max}, I^{\max}}^+ = \frac{R(\mathbf{z}_{I^{\max}})}{Q(\mathbf{z}_{I^{\max}})}.$$

For example, if $N = n = 2$ and $\lambda = (1, 1)$, then $\xi_{(1,2)}^+(z_1, z_2, h) = v_{(1,2)}$ and

$$\xi_{(2,1)}^+(z_1, z_2, h) = \frac{z_2 - z_1}{z_2 - z_1 + h} v_{(2,1)} + \frac{h}{z_2 - z_1 + h} v_{(1,2)}.$$

Proposition 2.15. *There exist unique elements $\{\xi_I^- \in V \otimes \mathbb{C}[z_1, \dots, z_n, h, D^{-1}] \mid I \in \mathcal{I}_\lambda\}$ such that $\xi_{I^{\max}}^- = v_{I^{\max}}$ and*

$$(2.19) \quad \xi_{s_i(I)}^- = \tilde{s}_i^- \xi_I^-$$

for every $I \in \mathcal{I}_\lambda$ and $i = 1, \dots, n-1$. Moreover,

$$(2.20) \quad \xi_I^- = \sum_{J \geq I} X_{I,J}^- v_J,$$

where $X_{I,J}^- \in \mathbb{C}[z_1, \dots, z_n, h, D^{-1}]$, $X_{I,I}^- \neq 0$, and

$$(2.21) \quad X_{I^{\min}, I^{\min}}^- = \frac{R(\mathbf{z}_{I^{\min}})}{Q(\mathbf{z}_{I^{\min}})}.$$

Proof of Propositions 2.14, 2.15. Proposition 2.14 follows from formula (2.3) and Lemma 2.2. Notice that $\tilde{s}_i^+ v_{I^{\min}} = v_{I^{\min}}$ if and only if $s_i(I^{\min}) = I^{\min}$. Similarly, Proposition 2.15 follows from formula (2.3) and Lemma 2.3. \square

For example, if $N = n = 2$ and $\lambda = (1, 1)$, then $\xi_{(2,1)}^-(z_1, z_2, h) = v_{(2,1)}$ and

$$\xi_{(1,2)}^-(z_1, z_2, h) = \frac{z_1 - z_2}{z_1 - z_2 + h} v_{(1,2)} + \frac{h}{z_1 - z_2 + h} v_{(2,1)}.$$

Let $f_I(z_1, \dots, z_n, h)$, $I \in \mathcal{I}_\lambda$, be a collection of scalar functions.

Lemma 2.16. *The V -valued function $\sum_I f_I(z_1, \dots, z_n, h) \xi_I^+$ (resp. $\sum_I f_I(z_1, \dots, z_n, h) \xi_I^-$), is invariant with respect to the S_n^+ -action (resp. S_n^- -action) if and only if*

$$f_{\sigma(I)}(z_1, \dots, z_n, h) = f_I(z_{\sigma_1}, \dots, z_{\sigma_n}, h)$$

for any $I \in \mathcal{I}_\lambda$ and any $\sigma \in S_n$. \square

Proposition 2.17. *For any $f \in \mathbb{C}[z_1, \dots, z_n]^{S_{\lambda_1} \times \dots \times S_{\lambda_N}} \otimes \mathbb{C}[h]$, we have*

$$(2.22) \quad \vartheta_{\lambda}^+(f) = \sum_{I \in \mathcal{I}_{\lambda}} \frac{f(\mathbf{z}_I, h) Q(\mathbf{z}_I)}{R(\mathbf{z}_I)} \xi_I^+,$$

$$(2.23) \quad \vartheta_{\lambda}^-(f) = \sum_{I \in \mathcal{I}_{\lambda}} \frac{f(\mathbf{z}_I, h)}{R(\mathbf{z}_I)} \xi_I^+,$$

$$(2.24) \quad \vartheta_{\lambda}^-(f) = \sum_{I \in \mathcal{I}_{\lambda}} \frac{f(\mathbf{z}_I, h)}{R(\mathbf{z}_I)} \xi_I^-.$$

Proof. By formulae (2.17), (2.18), we have $\vartheta_{\lambda}^+(f) = \sum_I c_I \xi_I^+$ for some coefficients c_I , and $c_{I^{\max}} = f(\mathbf{z}_{I^{\max}}, h) Q(\mathbf{z}_{I^{\max}})/R(\mathbf{z}_{I^{\max}})$. Now formulae (2.22), (2.23) follow from Lemma 2.16.

Formula (2.24) follows similarly from formulae (2.20), (2.21), and Lemma 2.16. \square

Theorem 2.18. *For $I, J \in \mathcal{I}_{\lambda}$, we have*

$$(2.25) \quad \mathcal{S}(\xi_I^+, \xi_J^-) = \delta_{I,J} \frac{R(\mathbf{z}_I)}{Q(\mathbf{z}_I)}.$$

Proof. If $I = I^{\min}$ or $J = I^{\max}$, the claim follows from formulae (2.17)–(2.21). Then formula (2.25) extends to arbitrary $I, J \in \mathcal{I}_{\lambda}$ by properties (2.16), (2.19). \square

2.5. Distinguished elements v_{λ}^+ , v_{λ}^- , and v_{λ}^- . Let $v_{\lambda}^+ = \sum_{I \in \mathcal{I}_{\lambda}} v_I$. Clearly, $v_{\lambda}^+ \in \mathcal{V}^+ \cap \frac{1}{D} \mathcal{V}_{\lambda}^-$. Moreover, $v_{\lambda}^+ = \vartheta_{\lambda}^+(1)$, $v_{\lambda}^+ = \vartheta_{\lambda}^-(Q_{\lambda})$ and $v_{\lambda}^+ = \vartheta_{\lambda}^-(Q_{\lambda})$, where Q_{λ} is given by (2.7).

Denote $v_{\lambda}^- = \vartheta_{\lambda}^-(1) \in \frac{1}{D} \mathcal{V}_{\lambda}^-$ and $v_{\lambda}^- = \vartheta_{\lambda}^-(1) \in \frac{1}{D} \mathcal{V}_{\lambda}^-$. By Proposition 2.17 we have

$$(2.26) \quad v_{\lambda}^+ = \sum_{I \in \mathcal{I}_{\lambda}} \frac{Q(\mathbf{z}_I)}{R(\mathbf{z}_I)} \xi_I^+ = \sum_{I \in \mathcal{I}_{\lambda}} \frac{Q(\mathbf{z}_I)}{R(\mathbf{z}_I)} \xi_I^-,$$

$$(2.27) \quad v_{\lambda}^- = \sum_{I \in \mathcal{I}_{\lambda}} \frac{1}{R(\mathbf{z}_I)} \xi_I^+, \quad v_{\lambda}^- = \sum_{I \in \mathcal{I}_{\lambda}} \frac{1}{R(\mathbf{z}_I)} \xi_I^-.$$

Lemma 2.19. *The vector v_{λ}^+ belongs to the irreducible \mathfrak{gl}_N -submodule of V of highest weight $(n, 0, \dots, 0)$, generated by the vector $v_1 \otimes \dots \otimes v_1$. \square*

Lemma 2.20 ([RTVZ]). *For any i, j such that $\lambda_i \geq \lambda_j$, we have $e_{i,j} v_{\lambda}^- = 0$ and $e_{i,j} v_{\lambda}^- = 0$ with respect to the pointwise \mathfrak{gl}_N -action on V . \square*

Notice that the functions v_{λ}^- and v_{λ}^- under certain conditions become quantized conformal blocks and satisfy qKZ equations with respect to z_1, \dots, z_n , see [RTVZ], cf. [V, RV, RSV].

3. BETHE SUBALGEBRAS

3.1. Yangian $Y(\mathfrak{gl}_N)$. The Yangian $Y(\mathfrak{gl}_N)$ is the unital associative algebra with generators $T_{i,j}^{(s)}$ for $i, j = 1, \dots, N$, $s \in \mathbb{Z}_{>0}$, subject to relations

$$(3.1) \quad (u - v) [T_{i,j}(u), T_{k,l}(v)] = T_{k,j}(v) T_{i,l}(u) - T_{k,j}(u) T_{i,l}(v), \quad i, j, k, l = 1, \dots, N,$$

where

$$T_{i,j}(u) = \delta_{i,j} + \sum_{s=1}^{\infty} T_{i,j}^{\{s\}} u^{-s}.$$

Let $T(u) = \sum_{i,j=1}^N E_{i,j} \otimes T_{i,j}(u)$, where $E_{i,j} \in \text{End}(\mathbb{C}^N)$ is the image of $e_{i,j} \in \mathfrak{gl}_N$. Relations (3.1) can be written as the equality of series with coefficients in $\text{End}(\mathbb{C}^N \otimes \mathbb{C}^N) \otimes Y(\mathfrak{gl}_N)$:

$$(3.2) \quad (u - v + P^{(1,2)}) T^{(1)}(u) T^{(2)}(v) = T^{(2)}(v) T^{(1)}(u) (u - v + P^{(1,2)}),$$

where $P^{(1,2)}$ is the permutation of the \mathbb{C}^N factors, $T^{(1)}(u) = \sum_{i,j=1}^N E_{i,j} \otimes 1 \otimes T_{i,j}(u)$ and $T^{(2)}(u) = 1 \otimes T(u)$.

The Yangian $Y(\mathfrak{gl}_N)$ is a Hopf algebra with the coproduct $\Delta : Y(\mathfrak{gl}_N) \rightarrow Y(\mathfrak{gl}_N) \otimes Y(\mathfrak{gl}_N)$ given by $\Delta(T_{i,j}(u)) = \sum_{k=1}^N T_{k,j}(u) \otimes T_{i,k}(u)$ for $i, j = 1, \dots, N$. The Yangian $Y(\mathfrak{gl}_N)$ contains $U(\mathfrak{gl}_N)$ as a Hopf subalgebra, the embedding given by $e_{i,j} \mapsto T_{j,i}^{\{1\}}$.

Notice that $[T_{i,j}^{\{1\}}, T_{k,l}^{\{s\}}] = \delta_{i,l} T_{k,j}^{\{s\}} - \delta_{j,k} T_{i,l}^{\{s\}}$ for $i, j, k, l = 1, \dots, N$, $s \in \mathbb{Z}_{>0}$, which implies that the Yangian $Y(\mathfrak{gl}_N)$ is generated by the elements $T_{i,i+1}^{\{1\}}, T_{i+1,i}^{\{1\}}$, $i = 1, \dots, N-1$, and $T_{1,1}^{\{s\}}$, $s > 0$.

The assignment

$$(3.3) \quad \varpi : T_{i,j}^{\{s\}} \mapsto T_{j,i}^{\{s\}}, \quad i, j = 1, \dots, N, \quad s > 0,$$

defines a Hopf anti-automorphism of the Yangian $Y(\mathfrak{gl}_N)$.

More information on the Yangian $Y(\mathfrak{gl}_N)$ can be found in [M]. Notice that the series $T_{i,j}(u)$ here corresponds to the series $T_{j,i}(u)$ in [M].

3.2. Bethe algebra. For $p = 1, \dots, N$, $\mathbf{i} = \{1 \leq i_1 < \dots < i_p \leq N\}$, $\mathbf{j} = \{1 \leq j_1 < \dots < j_p \leq N\}$, define

$$M_{\mathbf{i},\mathbf{j}}(u) = \sum_{\sigma \in S_p} (-1)^\sigma T_{i_1, j_{\sigma(1)}}(u) \dots T_{i_p, j_{\sigma(p)}}(u - p + 1).$$

For $\mathbf{i} = \{1, \dots, N\}$, the series $M_{\mathbf{i},\mathbf{i}}(u)$ is called the quantum determinant and denoted by $\text{qdet } T(u)$. Its coefficients generate the center of the Yangian $Y(\mathfrak{gl}_N)$.

We have

$$(3.4) \quad \Delta(M_{\mathbf{i},\mathbf{j}}(u)) = \sum_{\mathbf{k} = \{1 \leq k_1 < \dots < k_p \leq N\}} M_{\mathbf{k},\mathbf{j}}(u) \otimes M_{\mathbf{i},\mathbf{k}}(u),$$

see, for example, [NT, Proposition 1.11] or [MTV1, Lemma 4.3]. Notice that the Yangian coproduct used here is opposite to that used in [NT]. We also have

$$(3.5) \quad \varpi(M_{\mathbf{i},\mathbf{j}}(u)) = M_{\mathbf{j},\mathbf{i}}(u),$$

see, for example, [NT, Lemma 1.5].

For $q = (q_1, \dots, q_N) \in (\mathbb{C}^*)^N$ and $p = 1, \dots, N$, define

$$(3.6) \quad B_p^q(u) = \sum_{i=\{1 \leq i_1 < \dots < i_p \leq N\}} q_{i_1} \dots q_{i_p} M_{i,i}(u) = \sigma_p(q_1, \dots, q_N) + \sum_{s=1}^{\infty} B_{p,s}^q u^{-s},$$

where σ_p is the p -th elementary symmetric function and $B_{p,s}^q \in Y(\mathfrak{gl}_N)$. Let $\mathcal{B}^q \subset Y(\mathfrak{gl}_N)$ be the unital subalgebra generated by the elements $B_{p,s}^q$, $p = 1, \dots, N$, $s \geq 0$. It is easy to see that the subalgebra \mathcal{B}^q does not change if all q_1, \dots, q_N are multiplied simultaneously by the same number. The algebra \mathcal{B}^q is called a *Bethe subalgebra* of $Y(\mathfrak{gl}_N)$.

Theorem 3.1 ([KS]). *The subalgebra \mathcal{B}^q is commutative and commutes with the subalgebra $U(\mathfrak{h}) \subset Y(\mathfrak{gl}_N)$.* \square

The series $B_p^q(u)$ depends polynomially on q_1, \dots, q_N . Let q tend to infinity so that $q_{i+1}/q_i \rightarrow 0$ for all $i = 1, \dots, N-1$, and $q_N = 1$. In this limit,

$$(3.7) \quad B_p^q(u) = q_1 \dots q_p (M_{i,i}(u) + o(1)), \quad i = \{1, \dots, p\}.$$

Introduce the elements $B_{p,s}^\infty \in Y(\mathfrak{gl}_N)$, $p = 1, \dots, N$, $s \in \mathbb{Z}_{>0}$, by the formula

$$(3.8) \quad M_{i,i}(u) = 1 + \sum_{s=1}^{\infty} B_{p,s}^\infty u^{-s}, \quad i = \{1, \dots, p\}.$$

Let $\mathcal{B}^\infty \subset Y(\mathfrak{gl}_N)$ be the unital subalgebra generated by the elements $B_{p,s}^\infty$, $p = 1, \dots, N$, $s > 0$. The algebra \mathcal{B}^∞ is called a *Gelfand-Zetlin subalgebra* of $Y(\mathfrak{gl}_N)$. It has been studied in [NT].

The subalgebra \mathcal{B}^∞ is commutative by Theorem 3.1. Since $B_{p,1}^\infty = T_{1,1}^{\{1\}} + \dots + T_{p,p}^{\{1\}}$ for $p = 1, \dots, N$, the subalgebra \mathcal{B}^∞ contains the subalgebra $U(\mathfrak{h}) \subset Y(\mathfrak{gl}_N)$.

Assume that q_1, \dots, q_N are distinct numbers. Define the elements $S_{i,s}^q \in \mathcal{B}^q$ for $i = 1, \dots, N$, $s \in \mathbb{Z}_{>0}$, by the rule

$$(3.9) \quad S_{i,s}^q = \sum_{p=1}^N (-1)^{p-1} B_{p,s}^q q_i^{N-p-1} \prod_{\substack{j=1 \\ j \neq i}}^N \frac{1}{q_i - q_j},$$

so that

$$\prod_{i=1}^N \frac{1}{x - q_i} \sum_{p=1}^N \sum_{s=1}^{\infty} (-1)^p B_p^q(u) x^{N-p} = 1 - \sum_{i=1}^N \sum_{s=1}^{\infty} \frac{q_i}{x - q_i} S_{i,s}^q u^{-s}.$$

In particular,

$$(3.10) \quad S_{i,1}^q = T_{i,i}^{\{1\}}, \quad S_{i,2}^q = T_{i,i}^{\{2\}} - \sum_{\substack{j=1 \\ j \neq i}}^N \frac{q_j}{q_i - q_j} (T_{i,i}^{\{1\}} T_{j,j}^{\{1\}} - T_{i,j}^{\{1\}} T_{j,i}^{\{1\}} + T_{j,j}^{\{1\}}).$$

Lemma 3.2. *For distinct numbers q_1, \dots, q_N , the subalgebra \mathcal{B}^q contains the subalgebra $U(\mathfrak{h}) \subset Y(\mathfrak{gl}_N)$.* \square

Lemma 3.3. *Let $q_{i+1}/q_i \rightarrow 0$ for all $i = 1, \dots, N-1$. Then $S_{i,s}^q \rightarrow B_{i,s}^\infty - B_{i-1,s}^\infty$ for $i = 1, \dots, N$, where $B_{0,s}^\infty = 0$.*

Proof. Write formula (3.9) as

$$S_{i,s}^q = \sum_{p=1}^N (-1)^{p-i} h^{s-1} B_{p,s}^q \frac{q_i^{i-p-1}}{q_1 \cdots q_{i-1}} \prod_{j=1}^{i-1} \frac{1}{1 - q_i/q_j} \prod_{j=i+1}^N \frac{1}{1 - q_j/q_i}.$$

Now the claim follows from formulae (3.7), (3.8). \square

Lemma 3.4. *For any $X \in \mathcal{B}^q$, we have $\varpi(X) = X$, and for any $X \in \mathcal{B}^\infty$, we have $\varpi(X) = X$.*

Proof. By formulae (3.5)–(3.8), see also [MTV1, Proposition 4.11], for any $p = 1, \dots, N$ and $s > 0$, we have $\varpi(B_{p,s}^q) = B_{p,s}^q$ and $\varpi(B_{p,s}^\infty) = B_{p,s}^\infty$. Since the algebras \mathcal{B}^q and \mathcal{B}^∞ are commutative, the statement follows. \square

As a subalgebra of $Y(\mathfrak{gl}_N)$, the Bethe algebra \mathcal{B}^q acts on any $Y(\mathfrak{gl}_N)$ -module M . Since \mathcal{B}^q commutes with $U(\mathfrak{h})$, it preserves the weight subspaces M_λ . If $L \subset M$ is a \mathcal{B}^q -invariant subspace, then the image of \mathcal{B}^q in $\text{End}(L)$ will be called the Bethe algebra of L and denoted by $\mathcal{B}^q(L)$.

3.3. Algebras $\tilde{Y}(\mathfrak{gl}_N)$, $\tilde{\mathcal{B}}^q$, $\tilde{\mathcal{B}}^\infty$. Let $\tilde{Y}(\mathfrak{gl}_N)$ be the subalgebra of $Y(\mathfrak{gl}_N) \otimes \mathbb{C}[h]$ generated over \mathbb{C} by $\mathbb{C}[h]$ and the elements $h^{s-1} T_{i,j}^{\{s\}}$ for $i, j = 1, \dots, N$, $s > 0$. Equivalently, the subalgebra $\tilde{Y}(\mathfrak{gl}_N)$ is generated over \mathbb{C} by $\mathbb{C}[h]$ and the elements $T_{i,i+1}^{\{1\}}, T_{i+1,i}^{\{1\}}$, $i = 1, \dots, N-1$, and $h^{s-1} T_{1,1}^{\{s\}}$, $s > 0$.

Let $\tilde{\mathcal{B}}^q$ (resp. $\tilde{\mathcal{B}}^\infty$) be the subalgebra of $Y(\mathfrak{gl}_N) \otimes \mathbb{C}[h]$ generated over \mathbb{C} by $\mathbb{C}[h]$ and the elements $h^{s-1} B_{p,s}^q$ (resp. $h^{s-1} B_{p,s}^\infty$) for $p = 1, \dots, N$, $s > 0$. It is easy to see that $\tilde{\mathcal{B}}^q$ and $\tilde{\mathcal{B}}^\infty$ are subalgebras of $\tilde{Y}(\mathfrak{gl}_N)$.

Introduce the series $A_1(u), \dots, A_N(u)$, $E_1(u), \dots, E_{N-1}(u)$, $F_1(u), \dots, F_{N-1}(u)$:

$$(3.11) \quad A_p(u) = M_{\mathbf{i}, \mathbf{i}}(u/h) = 1 + \sum_{s=1}^{\infty} h^s B_{p,s}^\infty u^{-s},$$

$$(3.12) \quad E_p(u) = h^{-1} M_{\mathbf{j}, \mathbf{i}}(u/h) (M_{\mathbf{i}, \mathbf{i}}(u/h))^{-1} = \sum_{s=1}^{\infty} h^{s-1} E_{p,s} u^{-s},$$

$$F_p(u) = h^{-1} (M_{\mathbf{i}, \mathbf{i}}(u/h))^{-1} M_{\mathbf{i}, \mathbf{j}}(u/h) = \sum_{s=1}^{\infty} h^{s-1} F_{p,s} u^{-s},$$

where $\mathbf{i} = \{1, \dots, p\}$, $\mathbf{j} = \{1, \dots, p-1, p+1\}$. Observe that $E_{p,1} = T_{p+1,p}^{\{1\}}$, $F_{p,1} = T_{p,p+1}^{\{1\}}$ and $B_{1,s}^\infty = T_{1,1}^{\{s\}}$, so the coefficients of the series $E_p(u)$, $F_p(u)$ and $h^{-1}(A_p(u) - 1)$ together with $\mathbb{C}[h]$ generate $\tilde{Y}(\mathfrak{gl}_N)$.

In what follows we will describe actions of the algebra $\tilde{Y}(\mathfrak{gl}_N)$ by using series (3.11), (3.12).

The quotient algebra $\tilde{Y}(\mathfrak{gl}_N)/(h-1)\tilde{Y}(\mathfrak{gl}_N)$ is canonically isomorphic to $Y(\mathfrak{gl}_N)$. Also, the quotient algebra $\tilde{Y}(\mathfrak{gl}_N)/h\tilde{Y}(\mathfrak{gl}_N)$ is naturally isomorphic to the universal enveloping algebra $U(\mathfrak{gl}_N[t])$, the element $h^{s-1} T_{i,j}^{\{s\}}$ projecting to $e_{j,i} \otimes t^{s-1}$.

3.4. Dynamical Hamiltonians and dynamical connection. Assume that q_1, \dots, q_N are distinct numbers. Define the elements $X_1^q, \dots, X_N^q \in \tilde{\mathcal{B}}^q$ by the rule

$$(3.13) \quad \begin{aligned} X_i^q &= hS_{i,2}^q - \frac{h}{2} T_{i,i}^{\{1\}} (T_{i,i}^{\{1\}} - 1) + h \sum_{\substack{j=1 \\ j \neq i}}^N \frac{q_j}{q_i - q_j} T_{i,i}^{\{1\}} T_{j,j}^{\{1\}} \\ &= hT_{i,i}^{\{2\}} - \frac{h}{2} T_{i,i}^{\{1\}} (T_{i,i}^{\{1\}} - 1) + h \sum_{\substack{j=1 \\ j \neq i}}^N \frac{q_j}{q_i - q_j} G_{i,j}, \end{aligned}$$

where $S_{i,2}^q$ is given by (3.10) and $G_{i,j} = T_{i,j}^{\{1\}} T_{j,i}^{\{1\}} - T_{j,j}^{\{1\}} = T_{j,i}^{\{1\}} T_{i,j}^{\{1\}} - T_{i,i}^{\{1\}}$. Here the second formula is obtained from the first one by using commutation relations of the elements $T_{k,l}^{\{1\}}$. Notice that $T_{i,i}^{\{1\}}$ and $G_{i,j}$ belong to the subalgebra $U(\mathfrak{gl}_N)$ embedded in $\tilde{Y}(\mathfrak{gl}_N)$,

$$T_{i,i}^{\{1\}} = e_{i,i}, \quad G_{i,j} = e_{i,j} e_{j,i} - e_{i,i} = e_{j,i} e_{i,j} - e_{j,j}.$$

Taking the limit $q_{i+1}/q_i \rightarrow 0$ for all $i = 1, \dots, N-1$, we define the elements $X_1^\infty, \dots, X_N^\infty \in \tilde{\mathcal{B}}^\infty$,

$$(3.14) \quad \begin{aligned} X_i^\infty &= hB_{i,2}^\infty - hB_{i-1,2}^\infty - \frac{h}{2} e_{i,i} (e_{i,i} - 1) - h e_{i,i} (e_{1,1} + \dots + e_{i-1,i-1}) \\ &= hT_{i,i}^{\{2\}} - \frac{h}{2} e_{i,i} (e_{i,i} - 1) - h(G_{i,1} + \dots + G_{i,i-1}). \end{aligned}$$

cf. Lemma 3.3. We will call the elements X_i^q, X_i^∞ , $i = 1, \dots, N$, the *dynamical Hamiltonians*. Observe that

$$(3.15) \quad X_i^q = X_i^\infty + h \sum_{j=1}^{i-1} \frac{q_i}{q_i - q_j} G_{i,j} + h \sum_{j=i+1}^N \frac{q_j}{q_i - q_j} G_{i,j}.$$

Let $G_{i,j}^+ = G_{i,j} - e_{i,i} e_{j,j}$. Given $\lambda = (\lambda_1, \dots, \lambda_N)$, set $G_{\lambda,i,j}^- = e_{j,i} e_{i,j}$ for $\lambda_i \geq \lambda_j$. and $G_{\lambda,i,j}^- = e_{i,j} e_{j,i}$ for $\lambda_i < \lambda_j$. Define the elements $X_1^{q+}, \dots, X_N^{q+}, X_{\lambda,1}^{q-}, \dots, X_{\lambda,N}^{q-} \in \tilde{\mathcal{B}}^q$,

$$(3.16) \quad X_i^{q+} = X_i^\infty + h \sum_{j=1}^{i-1} \frac{q_i}{q_i - q_j} G_{i,j}^+ + h \sum_{j=i+1}^N \frac{q_j}{q_i - q_j} G_{i,j}^+,$$

$$(3.17) \quad X_{\lambda,i}^{q-} = X_i^\infty + h \sum_{j=1}^{i-1} \frac{q_i}{q_i - q_j} G_{\lambda,i,j}^- + h \sum_{j=i+1}^N \frac{q_j}{q_i - q_j} G_{\lambda,i,j}^-.$$

It is straightforward to verify that for any nonzero complex number κ , the formal differential operators

$$(3.18) \quad \nabla_i = \kappa q_i \frac{\partial}{\partial q_i} - X_i^q, \quad i = 1, \dots, N,$$

pairwise commute and, hence, define a flat connection for any $\tilde{Y}(\mathfrak{gl}_N)$ -module.

Lemma 3.5. *The connections ∇^+ and ∇_λ^- defined by*

$$(3.19) \quad \nabla_i^+ = \kappa q_i \frac{\partial}{\partial q_i} - X_i^{q+}, \quad \nabla_{\lambda,i}^- = \kappa q_i \frac{\partial}{\partial q_i} - X_{\lambda,i}^{q-},$$

$i = 1, \dots, N$, are flat for any κ .

Proof. The connections ∇^+ and ∇_λ^- are gauge equivalent to connection (3.18),

$$(3.20) \quad \begin{aligned} \nabla_i^+ &= (\Theta^+)^{-1} \nabla_i \Theta^+, & \Theta^+ &= \prod_{1 \leq i < j \leq N} (1 - q_j/q_i)^{h e_{i,i} e_{j,j}/\kappa}, \\ \nabla_{\lambda,i}^- &= (\Theta_\lambda^-)^{-1} \nabla_i \Theta_\lambda^-, & \Theta_\lambda^- &= \prod_{1 \leq i < j \leq N} (1 - q_j/q_i)^{-h \varepsilon_{\lambda,i,j}/\kappa}, \end{aligned}$$

where $\varepsilon_{\lambda,i,j} = e_{j,j}$ for $\lambda_i \geq \lambda_j$, and $\varepsilon_{\lambda,i,j} = e_{i,i}$ for $\lambda_i < \lambda_j$. □

Connection (3.18) was introduced in [TV1], see also [MTV1], and is called the *trigonometric dynamical connection*. The connection is defined over \mathbb{C}^N with coordinates q_1, \dots, q_N , and has singularities at the union of the diagonals $q_i = q_j$. In the case of a tensor product of evaluation $\tilde{Y}(\mathfrak{gl}_N)$ -modules, the trigonometric dynamical connection commutes with the associated qKZ difference connection, see [TV1]. Under the $(\mathfrak{gl}_N, \mathfrak{gl}_n)$ duality, the trigonometric dynamical connection and the associated qKZ difference connection are respectively identified with the trigonometric KZ connection and the dynamical difference connection, see [TV1]. Notice that the trigonometric dynamical connection here differs from that in [TV1] by a gauge transformation.

4. YANGIAN ACTIONS

4.1. $\tilde{Y}(\mathfrak{gl}_N)$ -actions. Let $\mathbb{C}[z_1, \dots, z_n, h]$ act on $V \otimes \mathbb{C}[z_1, \dots, z_n, h]$ by multiplication on the second factor. Set

$$(4.1) \quad \begin{aligned} L^+(u) &= (u - z_n + hP^{(0,n)}) \dots (u - z_1 + hP^{(0,1)}), \\ L^-(u) &= (u - z_1 + hP^{(0,1)}) \dots (u - z_n + hP^{(0,n)}), \end{aligned}$$

where the factors of $\mathbb{C}^N \otimes V$ are labeled by $0, 1, \dots, n$. Both $L^+(u)$ and $L^-(u)$ are polynomials in u, z_1, \dots, z_n, h with values in $\text{End}(\mathbb{C}^N \otimes V)$. We consider $L^\pm(u)$ as $N \times N$ matrices with $\text{End}(V) \otimes \mathbb{C}[u, z_1, \dots, z_n, h]$ -valued entries $L_{i,j}^\pm(u)$.

Proposition 4.1. *The assignments*

$$(4.2) \quad \phi^\pm(T_{i,j}(u/h)) = L_{i,j}^\pm(u) \prod_{a=1}^n (u - z_a)^{-1}$$

define the actions ϕ^+ and ϕ^- of the algebra $\tilde{Y}(\mathfrak{gl}_N)$ on $V \otimes \mathbb{C}[z_1, \dots, z_n, h]$. Here the right-hand side of (4.2) is a series in u^{-1} with coefficients in $\text{End}(V) \otimes \mathbb{C}[z_1, \dots, z_n, h]$.

Proof. The claim follows from relations (3.2) and the Yang-Baxter equation

$$(4.3) \quad \begin{aligned} (u - v + hP^{(1,2)})(u + hP^{(1,3)})(v + hP^{(2,3)}) &= \\ &= (v + hP^{(2,3)})(u + hP^{(1,3)})(u - v + hP^{(1,2)}). \end{aligned} \quad \square$$

For both actions ϕ^\pm , the subalgebra $U(\mathfrak{gl}_N) \subset \tilde{Y}(\mathfrak{gl}_N)$ acts on $V \otimes \mathbb{C}[z_1, \dots, z_n, h]$ in the standard way: any element $x \in \mathfrak{gl}_N$ acts as $x^{(1)} + \dots + x^{(n)}$. The actions ϕ^\pm clearly commute with the $\mathbb{C}[z_1, \dots, z_n, h]$ -action.

Lemma 4.2. *The Yangian actions ϕ^\pm are contravariantly related through the Shapovalov pairing (2.11):*

$$\mathcal{S}(\phi^+(X)f, g) = \mathcal{S}(f, \phi^-(\varpi(X))g),$$

for any $X \in \tilde{Y}(\mathfrak{gl}_N)$ and any V -valued functions f, g of z_1, \dots, z_n, h .

Proof. The claim follows from formulae (3.3), (4.1) and (4.2). \square

Corollary 4.3. *For any $X \in \tilde{\mathcal{B}}^q$ or $X \in \tilde{\mathcal{B}}^\infty$, and any V -valued functions f, g of z_1, \dots, z_n, h , we have*

$$\mathcal{S}(\phi^+(X)f, g) = \mathcal{S}(f, \phi^-(X)g)$$

Proof. The claim follows from Lemma 3.4. \square

Lemma 4.4. *The Yangian action ϕ^+ (resp. ϕ^-) on V -valued functions of z_1, \dots, z_n, h commutes with the S_n^+ -action (2.1) (resp. the S_n^- -action (2.2)).*

Proof. The claim follows from the Yang-Baxter equation (4.3) and the fact that the actions ϕ^\pm commute with multiplication by functions of z_1, \dots, z_n, h . \square

By Lemma 4.4, the action ϕ^+ makes the spaces \mathcal{V}^+ and $\frac{1}{D}\mathcal{V}^=$ into $\tilde{Y}(\mathfrak{gl}_N)$ -modules, and the action ϕ^- on $V \otimes \mathbb{C}[z_1, \dots, z_n, h]$ makes the space $\frac{1}{D}\mathcal{V}^-$ into a $\tilde{Y}(\mathfrak{gl}_N)$ -modules.

Let $\Pi \in \text{End}(V)$ be the following linear map, $\Pi(g_1 \otimes \dots \otimes g_n) = g_n \otimes \dots \otimes g_1$ for any $g_1, \dots, g_n \in \mathbb{C}^N$. For any V -valued function $f(z_1, \dots, z_n, h)$ set

$$(4.4) \quad \tilde{\Pi}f(z_1, \dots, z_n, h) = \Pi(f(z_n, \dots, z_1, h)).$$

Lemma 4.5. *The map (4.4) induces an isomorphism $\frac{1}{D}\mathcal{V}^= \rightarrow \frac{1}{D}\mathcal{V}^-$ of $\tilde{Y}(\mathfrak{gl}_N)$ -modules. Moreover, $\tilde{\Pi}v_{\bar{\lambda}} = v_{\bar{\lambda}}$ for any λ .*

Proof. We have $\tilde{\Pi}\tilde{s}_i^+ = \tilde{s}_{n-i}^-\tilde{\Pi}$ for every $i = 1, \dots, n-1$, where the operators \tilde{s}_i^\pm are given by (2.3). Thus by Lemma 2.4, $\tilde{\Pi}$ induces a vector space isomorphism $\frac{1}{D}\mathcal{V}^= \rightarrow \frac{1}{D}\mathcal{V}^-$. Since

$$(4.5) \quad \tilde{\Pi}\phi^+(X) = \phi^-(X)\tilde{\Pi}$$

for every $X \in \tilde{Y}(\mathfrak{gl}_N)$, see (4.1), (4.2), $\tilde{\Pi}$ induces an isomorphism of $\tilde{Y}(\mathfrak{gl}_N)$ -modules. The equality $\tilde{\Pi}v_{\bar{\lambda}} = v_{\bar{\lambda}}$ follows from formulae (2.9), (2.10). \square

Proposition 4.6. *The $\tilde{Y}(\mathfrak{gl}_N)$ -module \mathcal{V}^+ is generated by the vector $v_1 \otimes \dots \otimes v_1$.*

Proof. For every $\lambda = (\lambda_1, \dots, \lambda_N)$, the subspace \mathcal{V}_λ^+ is generated by the $\tilde{\mathcal{B}}^\infty$ -action on v_λ^+ , see Theorem 5.4 in Section 5.3. Since

$$v_\lambda^+ = \phi^+ \left((T_{2,1}^{\{1\}})^{\lambda_1 - \lambda_2} \dots (T_{N-1,1}^{\{1\}})^{\lambda_{N-1} - \lambda_N} (T_{N,1}^{\{1\}})^{\lambda_N} \right) \frac{v_1 \otimes \dots \otimes v_1}{(\lambda_1 - \lambda_2)! \dots (\lambda_{N-1} - \lambda_N)! \lambda_N!},$$

the $\tilde{Y}(\mathfrak{gl}_N)$ -module \mathcal{V}^+ is generated by $v_1 \otimes \dots \otimes v_1$. \square

Notice that the $\tilde{Y}(\mathfrak{gl}_N)$ -modules \mathcal{V}^+ is not isomorphic to $\frac{1}{D}\mathcal{V}^\pm \simeq \frac{1}{D}\mathcal{V}^-$ because for any homomorphism $\mathcal{V}^+ \rightarrow \frac{1}{D}\mathcal{V}^\pm$ of $\tilde{Y}(\mathfrak{gl}_N)$ -modules, the image of the vector $v_1 \otimes \dots \otimes v_1$ belongs to $(v_1 \otimes \dots \otimes v_1) \otimes \mathbb{C}[z_1, \dots, z_n]^{S_n} \otimes \mathbb{C}[h]$ by the weight reasoning and generates a proper $\tilde{Y}(\mathfrak{gl}_N)$ -submodule of $\frac{1}{D}\mathcal{V}^\pm$.

Write $n = kN + l$ for $k, l \in \mathbb{Z}_{\geq 0}$, $l < N$. Set $\mu = (k+1, \dots, k+1, k, \dots, k)$ with l parts equal to $k+1$ and $N-l$ parts equal to k .

Proposition 4.7. *The $\tilde{Y}(\mathfrak{gl}_N)$ -module $\frac{1}{D}\mathcal{V}^-$ (resp. $\frac{1}{D}\mathcal{V}^\pm$) is generated by the function v_μ^- (resp. v_μ^\pm).*

Proof. For every $\lambda = (\lambda_1, \dots, \lambda_N)$, the subspace $\frac{1}{D}\mathcal{V}_\lambda^-$ is generated by the $\tilde{\mathcal{B}}^\infty$ -action on v_λ^- , see Theorem 5.6 in Section 5.3. By formula (4.11) below, we have

$$(4.6) \quad \phi^-(h^{\lambda_p - \lambda_{p+1} + 1} E_{p, \lambda_p - \lambda_{p+1} + 2}) v_\lambda^- = (-1)^{\lambda_p} v_{\lambda - \alpha_p}^-,$$

if $\lambda_p - \lambda_{p+1} \geq -1$. Here $\alpha_p = (0, \dots, 0, 1, -1, 0, \dots, 0)$, with $p-1$ first zeros. Similarly, by formula (4.12) below, we have

$$\phi^-(h^{\lambda_{p+1} - \lambda_p + 1} F_{p, \lambda_{p+1} - \lambda_p + 2}) v_\lambda^- = (-1)^{\lambda_p - 1} v_{\lambda + \alpha_p}^-,$$

if $\lambda_{p+1} - \lambda_p \geq -1$. Thus every v_λ^- can be obtained from v_μ^- by the $\tilde{Y}(\mathfrak{gl}_N)$ -action. This proves the statement for $\frac{1}{D}\mathcal{V}^-$. The proof for $\frac{1}{D}\mathcal{V}^\pm$ is similar. \square

Proof of Lemma 2.12. Clearly, if the statement holds for $\lambda = (\lambda_1, \dots, \lambda_N)$, then it holds for $\lambda^\sigma = (\lambda_{\sigma(1)}, \dots, \lambda_{\sigma(N)})$ for any $\sigma \in S_N$. So it suffices to prove the statement assuming that $\lambda_1 \geq \dots \geq \lambda_N$. Then by (4.6), there is $X \in \tilde{Y}(\mathfrak{gl}_N)$ such that $\phi^-(X) v_\lambda^- = v_1 \otimes \dots \otimes v_1$. Thus by Lemma 4.2,

$$\mathcal{S}(\phi^+(X) v_1 \otimes \dots \otimes v_1, v_\lambda^-) = \mathcal{S}(v_1 \otimes \dots \otimes v_1, \phi^-(X) v_\lambda^-) = 1. \quad \square$$

Proposition 4.8. *The Bethe algebras $\tilde{\mathcal{B}}^q(\mathcal{V}^+)$, $\tilde{\mathcal{B}}^q(\frac{1}{D}\mathcal{V}^\pm)$, and $\tilde{\mathcal{B}}^q(\frac{1}{D}\mathcal{V}^-)$ are isomorphic. Similarly, the Bethe algebras $\tilde{\mathcal{B}}^\infty(\mathcal{V}^+)$, $\tilde{\mathcal{B}}^\infty(\frac{1}{D}\mathcal{V}^\pm)$, and $\tilde{\mathcal{B}}^\infty(\frac{1}{D}\mathcal{V}^-)$ are isomorphic.*

Proof. The statement follows from Corollary 4.3 and Lemma 2.13. For the pairs $\tilde{\mathcal{B}}^q(\frac{1}{D}\mathcal{V}^\pm)$, $\tilde{\mathcal{B}}^q(\frac{1}{D}\mathcal{V}^-)$ and $\tilde{\mathcal{B}}^\infty(\frac{1}{D}\mathcal{V}^\pm)$, $\tilde{\mathcal{B}}^\infty(\frac{1}{D}\mathcal{V}^-)$, the algebras are isomorphic by Lemma 4.5 as well. \square

4.2. $Y(\mathfrak{gl}_N)$ -actions. Let $\phi_0^\pm : Y(\mathfrak{gl}_N) \rightarrow \text{End}(V)$ be the homomorphisms obtained from ϕ^\pm by taking $z_1 = \dots = z_n = 0$ and $h = 1$. They make the space V into $Y(\mathfrak{gl}_N)$ -modules, respectively denoted by $V^{(\pm)}$. The following statement is well known.

Proposition 4.9. *The $Y(\mathfrak{gl}_N)$ -modules $V^{(\pm)}$ are irreducible and isomorphic. The isomorphism $V^{(+)} \rightarrow V^{(-)}$ is given by the map $x_1 \otimes \dots \otimes x_n \mapsto x_n \otimes \dots \otimes x_1$ for any $x_1, \dots, x_n \in \mathbb{C}^N$. The modules $V^{(\pm)}$ are contravariantly dual,*

$$\mathcal{S}(\phi_0^+(X)f, g) = \mathcal{S}(f, \phi_0^-(\varpi(X))g),$$

for any $X \in Y(\mathfrak{gl}_N)$ and $f, g \in V$. \square

Let $\mathcal{J} \subset \mathbb{C}[z_1, \dots, z_n]^{S_n} \otimes \mathbb{C}[h]$ be the ideal generated by the relations $h = 1$ and $\sigma_i(z_1, \dots, z_n) = 0$ for all $i = 1, \dots, n$, where σ_i is the i -th elementary symmetric function. The quotient $\widehat{V} = V \otimes \mathbb{C}[z_1, \dots, z_n, h] / \mathcal{J}(V \otimes \mathbb{C}[z_1, \dots, z_n, h]) \simeq V \otimes (\mathbb{C}[z_1, \dots, z_n, h] / \mathcal{J})$ is a complex vector space of dimension $n! N^n$. The actions ϕ^\pm make it into $Y(\mathfrak{gl}_N)$ -modules respectively denoted by $\widehat{V}^{(\pm)}$.

Lemma 4.10. *The evaluation assignment*

$$(4.7) \quad f(z_1, \dots, z_n, h) \mapsto f(0, \dots, 0, 1)$$

defines homomorphisms $\widehat{V}^{(+)} \rightarrow V^{(+)}$ and $\widehat{V}^{(-)} \rightarrow V^{(-)}$ of $Y(\mathfrak{gl}_N)$ -modules. \square

Notice that the $Y(\mathfrak{gl}_N)$ -modules $\widehat{V}^{(\pm)}$ are respectively isomorphic to the $Y(\mathfrak{gl}_N)$ -modules $V^{(\pm)} \otimes (\mathbb{C}[z_1, \dots, z_n, h] / \mathcal{J})$ where the second factor $\mathbb{C}[z_1, \dots, z_n, h] / \mathcal{J}$ is the trivial $Y(\mathfrak{gl}_N)$ -module with all the generators $T_{i,j}^{\{s\}}$ acting by zero. However, the isomorphisms are not given by the identity operator on $V \otimes (\mathbb{C}[z_1, \dots, z_n, h] / \mathcal{J})$.

The quotients $\mathcal{V}^+ / \mathcal{J}\mathcal{V}^+$, $\frac{1}{D}\mathcal{V}^= / \mathcal{J}\frac{1}{D}\mathcal{V}^=$ and $\frac{1}{D}\mathcal{V}^- / \mathcal{J}\frac{1}{D}\mathcal{V}^-$ are complex vector spaces of dimension N^n . The $\widetilde{Y}(\mathfrak{gl}_N)$ -module structure on the respective spaces \mathcal{V}^+ , $\frac{1}{D}\mathcal{V}^=$, $\frac{1}{D}\mathcal{V}^-$ make $\mathcal{V}^+ / \mathcal{J}\mathcal{V}^+$, $\frac{1}{D}\mathcal{V}^= / \mathcal{J}\frac{1}{D}\mathcal{V}^=$ and $\frac{1}{D}\mathcal{V}^- / \mathcal{J}\frac{1}{D}\mathcal{V}^-$ into $Y(\mathfrak{gl}_N)$ -modules.

Theorem 4.11. *The assignment (4.7) defines isomorphisms*

$$\eta^+ : \mathcal{V}^+ / \mathcal{J}\mathcal{V}^+ \rightarrow V^{(+)}, \quad \eta^= : \frac{1}{D}\mathcal{V}^= / \mathcal{J}\frac{1}{D}\mathcal{V}^= \rightarrow V^{(+)}, \quad \eta^- : \frac{1}{D}\mathcal{V}^- / \mathcal{J}\frac{1}{D}\mathcal{V}^- \rightarrow V^{(-)}$$

of $Y(\mathfrak{gl}_N)$ -modules.

Proof. Clearly, the maps η^+ , $\eta^=$ and η^- are nonzero homomorphisms of $Y(\mathfrak{gl}_N)$ -modules, see Lemma 4.10. Since the $Y(\mathfrak{gl}_N)$ -modules $V^{(\pm)}$ are irreducible and all the spaces $\mathcal{V}^+ / \mathcal{J}\mathcal{V}^+$, $\frac{1}{D}\mathcal{V}^= / \mathcal{J}\frac{1}{D}\mathcal{V}^=$, $\frac{1}{D}\mathcal{V}^- / \mathcal{J}\frac{1}{D}\mathcal{V}^-$ and $V^{(\pm)}$ have the same dimension, the maps η^+ , $\eta^=$ and η^- are isomorphisms. \square

Corollary 4.12. *The Shapovalov pairing*

$$(4.8) \quad \mathcal{S} : \mathcal{V}^+ / \mathcal{J}\mathcal{V}^+ \otimes \frac{1}{D}\mathcal{V}^- / \mathcal{J}\frac{1}{D}\mathcal{V}^- \rightarrow (\mathbb{C}[z_1, \dots, z_n]^{S_n} \otimes \mathbb{C}[h]) / \mathcal{J} \simeq \mathbb{C}$$

induced by pairing (2.12) is nondegenerate. The $Y(\mathfrak{gl}_N)$ -modules $\mathcal{V}^+ / \mathcal{J}\mathcal{V}^+$ and $\frac{1}{D}\mathcal{V}^- / \mathcal{J}\frac{1}{D}\mathcal{V}^-$ are contravariantly dual:

$$\mathcal{S}(\phi^+(X)f, g) = \mathcal{S}(f, \phi^-(\varpi(X))g),$$

for any $X \in Y(\mathfrak{gl}_N)$ and $f \in \mathcal{V}^+/\mathcal{J}\mathcal{V}^+$, $g \in \frac{1}{D}\mathcal{V}^-/\mathcal{J}\frac{1}{D}\mathcal{V}^-$. \square

Alternatively, the nondegeneracy of pairing (4.8) follows from Lemma 2.11, Proposition 5.2 in Section 5 and the nondegeneracy of the Poincare pairing on \mathcal{F}_λ .

Corollary 4.13. *The Shapovalov pairing*

$$\mathcal{S} : \frac{1}{D}\mathcal{V}^=/\mathcal{J}\frac{1}{D}\mathcal{V}^= \otimes \frac{1}{D}\mathcal{V}^-/\mathcal{J}\frac{1}{D}\mathcal{V}^- \rightarrow (Z^{-1}\mathbb{C}[z_1, \dots, z_n]^{S_n} \otimes \mathbb{C}[h])/\mathcal{J} \simeq \mathbb{C}$$

induced by pairing (2.13) is nondegenerate. The $Y(\mathfrak{gl}_N)$ -modules $\frac{1}{D}\mathcal{V}^=/\mathcal{J}\frac{1}{D}\mathcal{V}^=$ and $\frac{1}{D}\mathcal{V}^-/\mathcal{J}\frac{1}{D}\mathcal{V}^-$ are contravariantly dual:

$$\mathcal{S}(\phi^+(X)f, g) = \mathcal{S}(f, \phi^-(\varpi(X))g),$$

for any $X \in Y(\mathfrak{gl}_N)$ and $f \in \frac{1}{D}\mathcal{V}^=/\mathcal{J}\frac{1}{D}\mathcal{V}^=$, $g \in \frac{1}{D}\mathcal{V}^-/\mathcal{J}\frac{1}{D}\mathcal{V}^-$. \square

4.3. Yangian actions on ξ_I^\pm . Denote

$$(4.9) \quad A_p^\pm(u) = \phi^\pm(A_p(u)), \quad E_p^\pm(u) = \phi^\pm(E_p(u)), \quad F_p^\pm(u) = \phi^\pm(F_p(u)).$$

Lemma 4.14. *The series $A_N^\pm(u)$ act on $V \otimes \mathbb{C}[z_1, \dots, z_n, h]$ as the operator of multiplication by $\prod_{j=1}^n (1 + h(u - z_j)^{-1})$.*

Proof. For $n = 1$, the proof is straightforward. For $n > 1$, the claim follows from the coproduct formula (3.4) and the $n = 1$ case. \square

Corollary 4.15. *Each of the images $\phi^\pm(\tilde{\mathcal{B}}^\infty)$ contains the subalgebra of multiplication operators by symmetric polynomials in variables z_1, \dots, z_n .*

Proof. The elements $\phi^\pm(h^{s-1}B_{N,s}^\infty) \in \text{End}(V) \otimes \mathbb{C}[z_1, \dots, z_n, h]$ have the form $\sum_{a=1}^n z_a^{s-1} + h g_{p,s}^\pm$, where $g_{p,s}^\pm \in \text{End}(V) \otimes \mathbb{C}[z_1, \dots, z_n, h]$ are symmetric in z_1, \dots, z_n and have homogeneous degree $s - 2$. The claim follows. \square

Theorem 4.16. *For each choice of the sign, $+$ or $-$, we have*

$$(4.10) \quad A_p^\pm(u) \xi_I^\pm = \xi_I^\pm \prod_{a=1}^p \prod_{i \in I_a} \left(1 + \frac{h}{u - z_i}\right),$$

$$(4.11) \quad E_p^\pm(u) \xi_I^\pm = \sum_{i \in I_{p+1}} \frac{\xi_{I^{i'}}^\pm}{u - z_i} \prod_{\substack{k \in I_{p+1} \\ k \neq i}} \frac{z_i - z_k + h}{z_i - z_k},$$

$$(4.12) \quad F_p^\pm(u) \xi_I^\pm = \sum_{j \in I_p} \frac{\xi_{I^{j'}}^\pm}{u - z_j} \prod_{\substack{k \in I_p \\ k \neq j}} \frac{z_j - z_k - h}{z_j - z_k},$$

where the sequences $I^{i'}$ and $I^{j'}$ are defined as follows: $I_a^{i'} = I_a^{j'} = I_a$ for $a \neq p, p+1$, and $I_p^{i'} = I_p \cup \{i\}$, $I_{p+1}^{i'} = I_{p+1} - \{i\}$, $I_p^{j'} = I_p - \{j\}$, $I_{p+1}^{j'} = I_{p+1} \cup \{j\}$.

Proof. Formulae (4.10)–(4.12) can be obtained from the results of [NT] as a particular case. Here we outline a partially alternative proof. We consider the case of the plus sign. The other case can be done similarly.

First observe that by formula (2.16) and Lemma 4.4, it suffices to prove formulae (4.10)–(4.12) only for $I = I^{\min}$. In this case, formula (4.10) for $n > 1$ follows from the coproduct formula (3.4) and the $n = 1$ case of (4.10). The proof of (4.10) for $n = 1$ is straightforward.

To get formula (4.11) for $I = I^{\min}$, observe that by formulae (2.17), (3.4), (4.10), we have $E_p^+(u) \xi_{I^{\min}}^+ = \sum_{i \in I_{p+1}} c_i \xi_{I^{\min}, i}^+$. The largest element of I_{p+1}^{\min} equals $i_{\max} = \lambda_1 + \dots + \lambda_{p+1}$, the coefficient $c_{i_{\max}}$ can be calculated due to the triangularity property (2.17), and $c_{i_{\max}}$ has the required form. The coefficient c_i for other $i \in I_{p+1}$ can be obtained from $c_{i_{\max}}$ by permuting z_i and $z_{i_{\max}}$ because I^{\min} is invariant under the transposition of i and i_{\max} . Thus all the coefficients c_i are as required, which proves formula (4.11).

The proof of formula (4.12) is similar. \square

4.4. Actions of the dynamical Hamiltonians.

Lemma 4.17. *For actions ϕ^\pm , see (4.2), of the dynamical Hamiltonians $X_1^\infty, \dots, X_N^\infty \in \widetilde{\mathcal{B}}^\infty$ on V -valued functions of z_1, \dots, z_n, h , we have*

$$(4.13) \quad \begin{aligned} \phi^+(X_i^\infty) &= \sum_{a=1}^n z_a e_{i,i}^{(a)} + \frac{h}{2} (e_{i,i} - e_{i,i}^2) + h \sum_{j=1}^N \sum_{1 \leq a < b \leq n} e_{i,j}^{(a)} e_{j,i}^{(b)} - h \sum_{j=1}^{i-1} G_{i,j}, \\ \phi^-(X_i^\infty) &= \sum_{a=1}^n z_a e_{i,i}^{(a)} + \frac{h}{2} (e_{i,i} - e_{i,i}^2) + h \sum_{j=1}^N \sum_{1 \leq b < a \leq n} e_{i,j}^{(a)} e_{j,i}^{(b)} - h \sum_{j=1}^{i-1} G_{i,j}, \end{aligned}$$

where $G_{i,j} = e_{i,j} e_{j,i} - e_{i,i} = e_{j,i} e_{i,j} - e_{j,j}$ and $e_{k,l} = e_{k,l}^{(1)} + \dots + e_{k,l}^{(n)}$ for every k, l .

Proof. The statement follow from formulae (4.1) and (3.13). \square

The operators $\phi^\pm(X_i^q)$, $\phi^\pm(X_i^{q+})$, $\phi^\pm(X_{\lambda,i}^{q-})$, $i = 1, \dots, N$, can be found from formulae (3.15)–(3.17).

Lemma 4.18. *For any λ and any $i = 1, \dots, N$, we have $\phi^+(X_i^{q+}) v_\lambda^+ = \phi^+(X_i^\infty) v_\lambda^+$,*

$$\phi^+(X_{\lambda,i}^{q-}) v_\lambda^- = \phi^+(X_i^\infty) v_\lambda^-, \quad \phi^-(X_{\lambda,i}^{q-}) v_\lambda^- = \phi^-(X_i^\infty) v_\lambda^-.$$

Proof. By Lemma 2.19, we have $(e_{i,i} e_{j,j} - e_{i,j} e_{j,i} + e_{i,i}) v_\lambda^+ = 0$ for any $i \neq j$, which yields the equality for v_λ^+ , see (3.16). By Lemma 2.20, we have $e_{i,j} v_\lambda^- = 0$ and $e_{i,j} v_\lambda^- = 0$ for any i, j such that $\lambda_i \geq \lambda_j$, which yields the equalities for v_λ^- and v_λ^- , see (3.17). \square

4.5. qKZ difference connection. Let

$$R^{(i,j)}(u) = \frac{u + hP^{(i,j)}}{u + h}, \quad i, j = 1, \dots, n, \quad i \neq j.$$

Define operators $K_1^\pm, \dots, K_n^\pm \in \text{End}(V) \otimes \mathbb{C}[z_1, \dots, z_n, h]$,

$$\begin{aligned} K_i^+ &= R^{(i,i-1)}(z_i - z_{i-1}) \dots R^{(i,1)}(z_i - z_1) \times \\ &\quad \times q_1^{e_{1,1}^{(i)}} \dots q_N^{e_{N,N}^{(i)}} R^{(i,n)}(z_i - z_n - \kappa) \dots R^{(i,i+1)}(z_i - z_{i+1} - \kappa), \\ K_i^- &= R^{(i,i+1)}(z_i - z_{i+1}) \dots R^{(i,n)}(z_i - z_n) \times \\ &\quad \times q_1^{e_{1,1}^{(i)}} \dots q_N^{e_{N,N}^{(i)}} R^{(i,1)}(z_i - z_1 - \kappa) \dots R^{(i,i-1)}(z_i - z_{i-1} - \kappa). \end{aligned}$$

Consider the difference operators $\widehat{K}_1^\pm, \dots, \widehat{K}_n^\pm$ acting on V -valued functions of z_1, \dots, z_n, h ,

$$\widehat{K}_i^\pm f(z_1, \dots, z_n, h) = K_i^\pm(z_1, \dots, z_n, h) f(z_1, \dots, z_{i-1}, z_i - \kappa, z_{i+1}, \dots, z_n).$$

Theorem 4.19 ([FR]). *The operators $\widehat{K}_1^+, \dots, \widehat{K}_n^+$ pairwise commute. Similarly, the operators $\widehat{K}_1^-, \dots, \widehat{K}_n^-$ pairwise commute.* \square

Theorem 4.20 ([TV1]). *The operators $\widehat{K}_1^+, \dots, \widehat{K}_n^+, \phi^+(\nabla_1^+), \dots, \phi^+(\nabla_N^+)$ pairwise commute. Similarly, the operators $\widehat{K}_1^-, \dots, \widehat{K}_n^-, \phi^-(\nabla_{\lambda,1}^-), \dots, \phi^-(\nabla_{\lambda,N}^-)$ pairwise commute.* \square

Corollary 4.21. *The operators $K_1^+|_{\kappa=0}, \dots, K_n^+|_{\kappa=0}, \phi^+(X_1^{q+}), \dots, \phi^+(X_N^{q+})$ pairwise commute. Similarly, the operators $K_1^-|_{\kappa=0}, \dots, K_n^-|_{\kappa=0}, \phi^-(X_{\lambda,1}^{q-}), \dots, \phi^-(X_{\lambda,N}^{q-})$ pairwise commute.*

The commuting difference operators $\widehat{K}_1^+, \dots, \widehat{K}_n^+$ define the qKZ difference connection. Similarly, the commuting difference operators $\widehat{K}_1^-, \dots, \widehat{K}_n^-$ define another qKZ difference connection. Theorem 4.20 says that the qKZ difference connections commute with the corresponding dynamical connections

Proposition 4.22. *For every $X \in \widetilde{\mathcal{B}}^q$, the operators $K_1^+|_{\kappa=0}, \dots, K_n^+|_{\kappa=0}$ commute with $\phi^+(X)$, and the operators $K_1^-|_{\kappa=0}, \dots, K_n^-|_{\kappa=0}$ commute with $\phi^-(X)$.*

Proof. By formulae (4.1), (4.2), (3.6), we have

$$K_i^\pm|_{\kappa=0} = \left(\phi^\pm(B_1^q(u)) \prod_{j=1}^n \frac{u - z_j}{u - z_j + h} \right) \Big|_{u=z_i}, \quad i = 1, \dots, n,$$

so the statement follows from Theorem 3.1. Alternatively, the proposition follows from the Yang-Baxter equation (4.3). \square

5. EQUIVARIANT COHOMOLOGY OF THE COTANGENT BUNDLES OF PARTIAL FLAG VARIETIES

5.1. Partial flag varieties. For $\lambda \in \mathbb{Z}_{\geq 0}^N$, $|\lambda| = n$, consider the partial flag variety \mathcal{F}_λ parametrizing chains of subspaces

$$0 = F_0 \subset F_1 \subset \dots \subset F_N = \mathbb{C}^n$$

with $\dim F_i/F_{i-1} = \lambda_i$, $i = 1, \dots, N$. Denote by $T^*\mathcal{F}_\lambda$ the cotangent bundle of \mathcal{F}_λ .

Let $T^n \subset GL_n = GL_n(\mathbb{C})$ be the torus of diagonal matrices. The groups $T^n \subset GL_n$ act on \mathbb{C}^n and hence on $T^*\mathcal{F}_\lambda$. Let the group \mathbb{C}^* act on $T^*\mathcal{F}_\lambda$ by multiplication in each fiber.

The set of fixed points $(T^*\mathcal{F}_\lambda)^{T^n \times \mathbb{C}^*}$ of the torus action lies in the zero section of the cotangent bundle and consists of the coordinate flags $F_I = (F_0 \subset \dots \subset F_N)$, $I = (I_1, \dots, I_N) \in \mathcal{I}_\lambda$, where F_i is the span of the standard basis vectors $u_j \in \mathbb{C}^n$ with $j \in I_1 \cup \dots \cup I_i$. The fixed points are in a one-to-one correspondence with the elements of \mathcal{I}_λ and hence with the basis vectors v_I of V_λ , see Section 2.1.

We consider the $GL_n(\mathbb{C}) \times \mathbb{C}^*$ -equivariant cohomology algebra

$$H_\lambda = H_{GL_n \times \mathbb{C}^*}^*(T^*\mathcal{F}_\lambda; \mathbb{C}).$$

Denote by $\Gamma_i = \{\gamma_{i,1}, \dots, \gamma_{i,\lambda_i}\}$ the set of the Chern roots of the bundle over \mathcal{F}_λ with fiber F_i/F_{i-1} . Let $\Gamma = (\Gamma_1; \dots; \Gamma_N)$. Denote by $\mathbf{z} = \{z_1, \dots, z_n\}$ the Chern roots corresponding to the factors of the torus T^n . Denote by h the Chern root corresponding to the factor \mathbb{C}^* acting on the fibers of $T^*\mathcal{F}_\lambda$ by multiplication. Then

$$(5.1) \quad H_\lambda = \mathbb{C}[\mathbf{z}; \Gamma]^{S_n \times S_{\lambda_1} \times \dots \times S_{\lambda_N}} \otimes \mathbb{C}[h] \left/ \left\langle \prod_{i=1}^N \prod_{j=1}^{\lambda_i} (u - \gamma_{i,j}) = \prod_{a=1}^n (u - z_a) \right\rangle \right.$$

The cohomology H_λ is a module over $H_{GL_n \times \mathbb{C}^*}^*(pt; \mathbb{C}) = \mathbb{C}[z_1, \dots, z_n]^{S_n} \otimes \mathbb{C}[h]$.

For $A \subset \{1, \dots, n\}$ denote $\mathbf{z}_A = \{z_a, a \in A\}$, and for $I = (I_1, \dots, I_N) \in \mathcal{I}_\lambda$ denote $\mathbf{z}_I = (z_{I_1}; \dots; z_{I_N})$. Set

$$(5.2) \quad \zeta_\lambda : H_\lambda \rightarrow \mathbb{C}[z_1, \dots, z_n]^{S_{\lambda_1} \times \dots \times S_{\lambda_N}} \otimes \mathbb{C}[h], \quad [f(\mathbf{z}; \Gamma; h)] \mapsto f(\mathbf{z}; \mathbf{z}_{I^{\min}}; h).$$

Clearly, ζ_λ is an isomorphism of $\mathbb{C}[z_1, \dots, z_n]^{S_n} \otimes \mathbb{C}[h]$ -algebras.

Let $\mathcal{J} \subset H_\lambda$ be the ideal generated by the relations $h = 1$ and $\sigma_i(z_1, \dots, z_n) = 0$, $i = 1, \dots, n$. Then $H_\lambda/\mathcal{J} = H^*(T^*\mathcal{F}_\lambda; \mathbb{C})$.

5.2. Equivariant integration. We will need the *integration on \mathcal{F}_λ map*

$$\int : H_\lambda \rightarrow H_{GL_n \times \mathbb{C}^*}^*(pt, \mathbb{C}),$$

that is, the composition

$$H_\lambda = H_{GL_n}^*(T^*\mathcal{F}_\lambda; \mathbb{C}) \otimes \mathbb{C}[h] \xrightarrow{f_{\mathcal{F}_\lambda} \otimes \text{id}} H_{GL_n}^*(pt; \mathbb{C}) \otimes \mathbb{C}[h] = H_{GL_n \times \mathbb{C}^*}^*(pt; \mathbb{C}).$$

The Atiyah-Bott equivariant localization theorem [AB] gives the integration on \mathcal{F}_λ map in terms of the fixed point set $(T^*\mathcal{F}_\lambda)^{T^n \times \mathbb{C}^*}$: for any $[f(\mathbf{z}; \Gamma; h)] \in H_\lambda$,

$$(5.3) \quad \int [f] = (-1)^{\sum_{i < j} \lambda_i \lambda_j} \sum_{I \in \mathcal{I}_\lambda} \frac{f(\mathbf{z}; \mathbf{z}_I; h)}{R(\mathbf{z}_I)},$$

where $R(\mathbf{z}_I)$ is given by (2.15). Clearly, the right-hand side in (5.3) lies in $\mathbb{C}[z_1, \dots, z_n]^{S_n} \otimes \mathbb{C}[h]$. The integration on \mathcal{F}_λ map induces the *Poincare pairing on \mathcal{F}_λ* ,

$$(5.4) \quad H_\lambda \otimes H_\lambda \rightarrow \mathbb{C}[z_1, \dots, z_n]^{S_n} \otimes \mathbb{C}[h], \quad [f] \otimes [g] \mapsto \int [fg].$$

After factorization by the ideal \mathcal{J} we obtain the nondegenerate Poincare pairing

$$(5.5) \quad H^*(T^*\mathcal{F}_\lambda; \mathbb{C}) \otimes H^*(T^*\mathcal{F}_\lambda; \mathbb{C}) \rightarrow \mathbb{C}.$$

We will also use the *integration on $T^*\mathcal{F}_\lambda$ map*,

$$(5.6) \quad \int : H^*(T^*\mathcal{F}_\lambda; \mathbb{C}) \rightarrow Z^{-1} \mathbb{C}[z_1, \dots, z_n]^{S_n} \otimes \mathbb{C}[h],$$

$$\int [f] = (-1)^{\sum_{i < j} \lambda_i \lambda_j} \sum_{I \in \mathcal{I}_\lambda} \frac{f(\mathbf{z}; \mathbf{z}_I; h)}{Q(\mathbf{z}_I) R(\mathbf{z}_I)}.$$

Here $Z = \prod_{i \neq j} (z_i - z_j + h)$, cf. (2.14). Notice that $Q(\mathbf{z}_I) R(\mathbf{z}_I)$ is the Euler class of the tangent space at the point $F_I \in T^*\mathcal{F}_\lambda$. This integration map was used in [BMO], see also [HP]. The integration on $T^*\mathcal{F}_\lambda$ map induces the *Poincare paring on $T^*\mathcal{F}_\lambda$* ,

$$(5.7) \quad H_\lambda \otimes H_\lambda \rightarrow Z^{-1} \mathbb{C}[z_1, \dots, z_n]^{S_n} \otimes \mathbb{C}[h], \quad [f] \otimes [g] \mapsto \int [fg],$$

which will be called the *Poincare paring on $T^*\mathcal{F}_\lambda$* .

Example. Let $N = n = 2$ and $\lambda = (1, 1)$. Then

$$\int [1] = 0, \quad \int [\gamma_{1,1}] = -1,$$

$$\int [1] = \frac{2}{(z_1 - z_2 + h)(z_2 - z_1 + h)}, \quad \int [\gamma_{1,1}] = \frac{z_1 + z_2 - h}{(z_1 - z_2 + h)(z_2 - z_1 + h)}.$$

5.3. H_λ and \mathcal{V}_λ^+ , $\frac{1}{D}\mathcal{V}_\lambda^-$, $\frac{1}{D}\mathcal{V}_\lambda^-$. Consider the maps $\nu_\lambda^+ : H_\lambda \rightarrow \mathcal{V}_\lambda^+$, $\nu_\lambda^- : H_\lambda \rightarrow \frac{1}{D}\mathcal{V}_\lambda^-$, and $\nu_\lambda^- : H_\lambda \rightarrow \frac{1}{D}\mathcal{V}_\lambda^-$,

$$(5.8) \quad \nu_\lambda^+ : [f] \mapsto \sum_{I \in \mathcal{I}_\lambda} \frac{f(\mathbf{z}; \mathbf{z}_I; h) Q(\mathbf{z}_I)}{R(\mathbf{z}_I)} \xi_I^+,$$

$$(5.9) \quad \nu_\lambda^- : [f] \mapsto \sum_{I \in \mathcal{I}_\lambda} \frac{f(\mathbf{z}; \mathbf{z}_I; h)}{R(\mathbf{z}_I)} \xi_I^+.$$

$$(5.10) \quad \nu_\lambda^- : [f] \mapsto \sum_{I \in \mathcal{I}_\lambda} \frac{f(\mathbf{z}; \mathbf{z}_I; h)}{R(\mathbf{z}_I)} \xi_I^-.$$

In particular, $\nu_\lambda^+ : [1] \mapsto v_\lambda^+$, $\nu_\lambda^- : [1] \mapsto v_\lambda^-$, $\nu_\lambda^- : [1] \mapsto v_\lambda^-$, see (2.26). We have $\nu_\lambda^+ = \vartheta_\lambda^+ \zeta_\lambda$, $\nu_\lambda^- = \vartheta_\lambda^- \zeta_\lambda$, $\nu_\lambda^- = \vartheta_\lambda^- \zeta_\lambda$, where the maps ϑ_λ^+ , ϑ_λ^- , ϑ_λ^- and ζ_λ are given by formulae (2.8)–(2.10) and (5.2). Observe that

$$(5.11) \quad \nu_\lambda^- = \tilde{\Pi} \nu_\lambda^-,$$

where $\tilde{\Pi}$ is given by formula (4.4).

Lemma 5.1. *The maps ν_λ^+ , ν_λ^- , ν_λ^- are isomorphisms of $\mathbb{C}[z_1, \dots, z_n]^{S_n} \otimes \mathbb{C}[h]$ -modules.*

Proof. The claim follows from Lemma 2.10. \square

Proposition 5.2. *The Shapovalov form and Poincare pairing on \mathcal{F}_λ are related by the formula*

$$\mathcal{S}(\nu_\lambda^+[f], \nu_\lambda^-[g]) = (-1)^{\sum_{i < j} \lambda_i \lambda_j} \int [f][g].$$

Proof. The statement follows from formulae (2.25), (5.3), (5.8), (5.10), and Lemma 5.1. \square

Proposition 5.3. *The Shapovalov form and Poincare pairing on $T^*\mathcal{F}_\lambda$ are related by the formula*

$$\mathcal{S}(\nu_\lambda^-[f], \nu_\lambda^-[g]) = (-1)^{\sum_{i < j} \lambda_i \lambda_j} \int [f][g].$$

Proof. The statement follows from formulae (2.25), (5.6), (5.9), (5.10), and Lemma 5.1. \square

Introduce the elements $f_{p,s} \in \mathbb{C}[\mathbf{z}; \Gamma]^{S_n \times S_{\lambda_1} \times \dots \times S_{\lambda_N}} \otimes \mathbb{C}[h]$, $p = 1, \dots, N$, $s \in \mathbb{Z}_{>0}$, by

$$(5.12) \quad \prod_{i=1}^{\lambda_p} \left(1 + \frac{h}{u - \gamma_{p,i}} \right) = 1 + h \sum_{s=1}^{\infty} f_{p,s} u^{-s}.$$

Since $f_{p,s} = \sum_{i=1}^{\lambda_p} \gamma_{p,i}^{s-1} + h g_{p,s}$, where $g_{p,s} \in \mathbb{C}[\mathbf{z}; \Gamma]^{S_n \times S_{\lambda_1} \times \dots \times S_{\lambda_N}} \otimes \mathbb{C}[h]$ are of homogeneous degree $s - 2$, the elements $[f_{p,s}]$ $p = 1, \dots, N$, $s > 0$, generate H_λ over $\mathbb{C}[h]$.

Define the elements $C_{p,s}^\pm \in \text{End}(V) \otimes \mathbb{C}[z_1, \dots, z_n, h]$, $p = 1, \dots, N$, $s \in \mathbb{Z}_{>0}$, by

$$(5.13) \quad (A_{p-1}^\pm(u))^{-1} A_p^\pm(u) = 1 + h \sum_{s=1}^{\infty} C_{p,s}^\pm u^{-s},$$

where $A_0^\pm(u) = 1$ and $A_1^\pm(u), \dots, A_N^\pm(u)$ are given by (4.9).

Let \mathcal{A} be a commutative algebra. The algebra \mathcal{A} considered as a module over itself is called the regular representation of \mathcal{A} .

Theorem 5.4. *The assignment $\mu_\lambda^+ : [f_{p,s}] \mapsto C_{p,s}^+$, $p = 1, \dots, N$, $s > 0$, extends uniquely to an isomorphism $H_\lambda \rightarrow \tilde{\mathcal{B}}^\infty(\mathcal{V}_\lambda^+)$ of $\mathbb{C}[z_1, \dots, z_n]^{S_n} \otimes \mathbb{C}[h]$ -algebras. The maps μ_λ^+ and ν_λ^+ identify the $\tilde{\mathcal{B}}^\infty(\mathcal{V}_\lambda^+)$ -module \mathcal{V}_λ^+ with the regular representation of the algebra H_λ .*

Theorem 5.5. *The assignment $\mu_\lambda^- : [f_{p,s}] \mapsto C_{p,s}^+$, $p = 1, \dots, N$, $s > 0$, extends uniquely to an isomorphism $H_\lambda \rightarrow \tilde{\mathcal{B}}^\infty(\frac{1}{D}\mathcal{V}_\lambda^-)$ of $\mathbb{C}[z_1, \dots, z_n]^{S_n} \otimes \mathbb{C}[h]$ -algebras. The maps μ_λ^- and ν_λ^- identify the $\tilde{\mathcal{B}}^\infty(\frac{1}{D}\mathcal{V}_\lambda^-)$ -module $\frac{1}{D}\mathcal{V}_\lambda^-$ with the regular representation of the algebra H_λ .*

Theorem 5.6. *The assignment $\mu_\lambda^- : [f_{p,s}] \mapsto C_{p,s}^-$, $p = 1, \dots, N$, $s > 0$, extends uniquely to an isomorphism $H_\lambda \rightarrow \tilde{\mathcal{B}}^\infty(\frac{1}{D}\mathcal{V}_\lambda^-)$ of $\mathbb{C}[z_1, \dots, z_n]^{S_n} \otimes \mathbb{C}[h]$ -algebras. The maps μ_λ^- and ν_λ^- identify the $\tilde{\mathcal{B}}^\infty(\frac{1}{D}\mathcal{V}_\lambda^-)$ -module $\frac{1}{D}\mathcal{V}_\lambda^-$ with the regular representation of the algebra H_λ .*

Proofs of Theorems 5.4 – 5.6. The operator $\nu_\lambda^+ C_{p,s}^+ (\nu_\lambda^+)^{-1}$ acts on H_λ as the multiplication by $[f_{p,s}]$, see Lemma 5.1 and formula (4.10). This yields Theorem 5.4. The proofs of Theorems 5.5 and 5.6 are similar. \square

Theorems 5.5 and 5.6 are equivalent by Lemma 4.5 and relation (5.11). In particular,

$$(5.14) \quad \tilde{\Pi} \mu_{\lambda}^{\bar{}}(f) = \mu_{\lambda}^{\bar{}}(f) \tilde{\Pi}$$

for any $f \in H_{\lambda}$, see (4.5).

Corollary 5.7. *The $\tilde{\mathcal{B}}^{\infty}$ -modules \mathcal{V}^+ , $\frac{1}{D}\mathcal{V}^=$, and $\frac{1}{D}\mathcal{V}^-$ are isomorphic.*

Proof. The isomorphisms restricted to weight subspaces are $\nu_{\lambda}^{\bar{}}(\nu_{\lambda}^+)^{-1}: \mathcal{V}_{\lambda}^+ \rightarrow \frac{1}{D}\mathcal{V}_{\lambda}^=$ and $\nu_{\lambda}^-(\nu_{\lambda}^+)^{-1}: \mathcal{V}_{\lambda}^+ \rightarrow \frac{1}{D}\mathcal{V}_{\lambda}^-$. \square

Corollary 5.8. *The subalgebras $\tilde{\mathcal{B}}^{\infty}(\mathcal{V}_{\lambda}^+) \subset \text{End}(\mathcal{V}_{\lambda}^+)$, $\tilde{\mathcal{B}}^{\infty}(\frac{1}{D}\mathcal{V}_{\lambda}^=) \subset \text{End}(\frac{1}{D}\mathcal{V}_{\lambda}^=)$ and $\tilde{\mathcal{B}}^{\infty}(\frac{1}{D}\mathcal{V}_{\lambda}^-) \subset \text{End}(\frac{1}{D}\mathcal{V}_{\lambda}^-)$ are maximal commutative subalgebras.* \square

Lemma 5.9. *For any $f \in H_{\lambda}$, and any $g_1 \in \mathcal{V}_{\lambda}^+$, $g_2 \in \frac{1}{D}\mathcal{V}_{\lambda}^=$, $g_3 \in \frac{1}{D}\mathcal{V}_{\lambda}^-$, we have*

$$\mathcal{S}(\mu_{\lambda}^+(f)g_1, g_3) = \mathcal{S}(g_1, \mu_{\lambda}^-(f)g_3), \quad \mathcal{S}(\mu_{\lambda}^{\bar{}}(f)g_2, g_3) = \mathcal{S}(g_2, \mu_{\lambda}^-(f)g_3).$$

Proof. The statement follows from the definition of the maps μ_{λ}^+ , $\mu_{\lambda}^{\bar{}}$, μ_{λ}^- and Corollary 4.3. \square

Example. Let $N = n = 2$ and $\lambda = (1, 1)$. Then the weight subspace V_{λ} is two-dimensional, $V_{\lambda} = \mathbb{C}v_{(1,2)} \oplus \mathbb{C}v_{(2,1)}$, $v_{(1,2)} = v_1 \otimes v_2$, $v_{(2,1)} = v_2 \otimes v_1$. The vectors ξ_I^{\pm} are

$$\begin{aligned} \xi_{(1,2)}^+ &= v_{(1,2)}, & \xi_{(2,1)}^+ &= \frac{z_2 - z_1}{z_2 - z_1 + h} v_{(2,1)} + \frac{h}{z_2 - z_1 + h} v_{(1,2)}, \\ \xi_{(2,1)}^- &= v_{(2,1)}, & \xi_{(1,2)}^- &= \frac{z_1 - z_2}{z_1 - z_2 + h} v_{(1,2)} + \frac{h}{z_1 - z_2 + h} v_{(2,1)}. \end{aligned}$$

Elements of \mathcal{V}_{λ}^+ have the form

$$f(z_1, z_2, h) \frac{z_1 - z_2 + h}{z_1 - z_2} \xi_{(1,2)}^+ + f(z_2, z_1, h) \frac{z_2 - z_1 + h}{z_2 - z_1} \xi_{(2,1)}^+,$$

elements of $\frac{1}{D}\mathcal{V}_{\lambda}^=$ have the form

$$\frac{f(z_1, z_2, h)}{z_1 - z_2} \xi_{(1,2)}^+ + \frac{f(z_2, z_1, h)}{z_2 - z_1} \xi_{(2,1)}^+,$$

and elements of $\frac{1}{D}\mathcal{V}_{\lambda}^-$ have the form

$$\frac{f(z_1, z_2, h)}{z_1 - z_2} \xi_{(1,2)}^- + \frac{f(z_2, z_1, h)}{z_2 - z_1} \xi_{(2,1)}^-.$$

The series (4.9), (5.13) for generators of the Bethe algebras $\tilde{\mathcal{B}}^{\infty}(\mathcal{V}_{\lambda}^+)$ and $\tilde{\mathcal{B}}^{\infty}(\frac{1}{D}\mathcal{V}_{\lambda}^-)$ are

$$\begin{aligned} A_2^{\pm}(u) &= \left(1 + \frac{h}{u - z_1}\right) \left(1 + \frac{h}{u - z_2}\right), \\ A_1^+(u) &= \begin{pmatrix} 1 + \frac{h}{u - z_1} & h^2 \\ 0 & 1 + \frac{h}{u - z_2} \end{pmatrix}, \quad (A_1^+(u))^{-1} A_2^+(u) = \begin{pmatrix} 1 + \frac{h}{u - z_2} & -h^2 \\ 0 & 1 + \frac{h}{u - z_1} \end{pmatrix}, \end{aligned}$$

$$A_1^-(u) = \begin{pmatrix} 1 + \frac{h}{u - z_1} & 0 \\ h^2 & 1 + \frac{h}{u - z_2} \end{pmatrix}, \quad (A_1^-(u))^{-1} A_2^-(u) = \begin{pmatrix} 1 + \frac{h}{u - z_2} & 0 \\ -h^2 & 1 + \frac{h}{u - z_1} \end{pmatrix},$$

where we are using the basis $v_{(1,2)}, v_{(2,1)}$ of V_λ . For the maps $\nu_\lambda^+, \nu_\lambda^-, \nu_\lambda^+$, and $\nu_\lambda^+, \nu_\lambda^-, \nu_\lambda^-$, we have

$$\begin{aligned} \mu_\lambda^+ : [1] &\mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mu_\lambda^+ : [\gamma_{1,1}] \mapsto \begin{pmatrix} z_1 & h \\ 0 & z_2 \end{pmatrix}, \\ \nu_\lambda^+ : [1] &\mapsto \frac{z_1 - z_2 + h}{z_1 - z_2} \xi_{(1,2)}^+ + \frac{z_2 - z_1 + h}{z_2 - z_1} \xi_{(2,1)}^+ = v_{(1,2)} + v_{(2,1)} = v_\lambda^+, \\ \nu_\lambda^+ : [\gamma_{1,1}] &\mapsto z_1 \frac{z_1 - z_2 + h}{z_1 - z_2} \xi_{(1,2)}^+ + z_2 \frac{z_2 - z_1 + h}{z_2 - z_1} \xi_{(2,1)}^+ = (z_1 + h) v_{(1,2)} + z_2 v_{(2,1)}, \\ \mu_\lambda^- : [1] &\mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mu_\lambda^- : [\gamma_{1,1}] \mapsto \begin{pmatrix} z_1 & h \\ 0 & z_2 \end{pmatrix}, \\ \nu_\lambda^- : [1] &\mapsto \frac{1}{z_1 - z_2} \xi_{(1,2)}^+ + \frac{1}{z_2 - z_1} \xi_{(2,1)}^+ = \frac{v_{(1,2)} - v_{(2,1)}}{z_1 - z_2 - h} = v_\lambda^-, \\ \nu_\lambda^- : [\gamma_{1,1}] &\mapsto \frac{z_1}{z_1 - z_2} \xi_{(1,2)}^+ + \frac{z_2}{z_2 - z_1} \xi_{(2,1)}^+ = \frac{(z_1 - h) v_{(1,2)} - z_2 v_{(2,1)}}{z_1 - z_2 - h}, \\ \mu_\lambda^- : [1] &\mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mu_\lambda^- : [\gamma_{1,1}] \mapsto \begin{pmatrix} z_1 & 0 \\ h & z_2 \end{pmatrix}, \\ \nu_\lambda^- : [1] &\mapsto \frac{1}{z_1 - z_2} \xi_{(1,2)}^- + \frac{1}{z_2 - z_1} \xi_{(2,1)}^- = \frac{v_{(1,2)} - v_{(2,1)}}{z_1 - z_2 + h} = v_\lambda^-, \\ \nu_\lambda^- : [\gamma_{1,1}] &\mapsto \frac{z_1}{z_1 - z_2} \xi_{(1,2)}^- + \frac{z_2}{z_2 - z_1} \xi_{(2,1)}^- = \frac{z_1 v_{(1,2)} - (z_2 - h) v_{(2,1)}}{z_1 - z_2 + h}, \end{aligned}$$

5.4. Cohomology as $\tilde{Y}(\mathfrak{gl}_N)$ -modules. The isomorphisms

$$\nu^+ = \bigoplus_\lambda \nu_\lambda^+ : \bigoplus_\lambda H_\lambda \rightarrow \mathcal{V}^+, \quad \nu^- = \bigoplus_\lambda \nu_\lambda^- : \bigoplus_\lambda H_\lambda \rightarrow \frac{1}{D} \mathcal{V}^-$$

induce two $\tilde{Y}(\mathfrak{gl}_N)$ -module structures on $\bigoplus_\lambda H_\lambda$ denoted respectively by ρ^+ and ρ^- ,

$$\rho^\pm(X) = (\nu^\pm)^{-1} \phi^\pm(X) \nu^\pm$$

for any $X \in \tilde{Y}(\mathfrak{gl}_N)$. The $\tilde{Y}(\mathfrak{gl}_N)$ -module structure on $\bigoplus_\lambda H_\lambda$ induced by the isomorphism

$$\nu^= = \bigoplus_\lambda \nu_\lambda^= : \bigoplus_\lambda H_\lambda \rightarrow \frac{1}{D} \mathcal{V}^=$$

coincides with the ρ^- -structure

$$(5.15) \quad (\nu^=)^{-1} \phi^+(X) \nu^= = \rho^-(X),$$

since $\nu_{\lambda}^- = \tilde{\Pi} \nu_{\lambda}^-$ and $\tilde{\Pi} \phi^+(X) = \phi^-(X) \tilde{\Pi}$, see (5.11), (4.5).

By formulae (5.8), (5.10) and (4.10), we have $\rho^{\pm}(A_p(u)) : H_{\lambda} \rightarrow H_{\lambda}$,

$$(5.16) \quad \rho^{\pm}(A_p(u)) : [f] \mapsto \left[f(\mathbf{z}; \Gamma; h) \prod_{a=1}^p \prod_{i=1}^{\lambda_p} \left(1 + \frac{h}{u - \gamma_{p,i}} \right) \right],$$

for $p = 1, \dots, N$. In particular, by (3.11), (3.14),

$$(5.17) \quad \rho^{\pm}(X_i^{\infty}) : [f] \mapsto [(\gamma_{i,1} + \dots + \gamma_{i,\lambda_i}) f(\mathbf{z}; \Gamma; h)], \quad i = 1, \dots, N.$$

Let $\alpha_1, \dots, \alpha_{N-1}$ be simple roots, $\alpha_p = (0, \dots, 0, 1, -1, 0, \dots, 0)$, with $p-1$ first zeros.

Theorem 5.10. *We have*

$$\rho^+(E_p(u)) : H_{\lambda - \alpha_p} \mapsto H_{\lambda},$$

$$\rho^+(E_p(u)) : [f] \mapsto \left[\sum_{i=1}^{\lambda_p} \frac{f(\mathbf{z}; \Gamma'^i; h)}{u - \gamma_{p,i}} \prod_{\substack{j=1 \\ j \neq i}}^{\lambda_p} \frac{\gamma_{p,i} - \gamma_{p,j} - h}{\gamma_{p,i} - \gamma_{p,j}} \right],$$

$$\rho^+(F_p(u)) : H_{\lambda + \alpha_p} \mapsto H_{\lambda},$$

$$\rho^+(F_p(u)) : [f] \mapsto \left[\sum_{i=1}^{\lambda_{p+1}} \frac{f(\mathbf{z}; \Gamma^{i'}; h)}{u - \gamma_{p+1,i}} \prod_{\substack{j=1 \\ j \neq i}}^{\lambda_{p+1}} \frac{\gamma_{p+1,i} - \gamma_{p+1,j} + h}{\gamma_{p+1,i} - \gamma_{p+1,j}} \right],$$

where

$$\Gamma'^i = (\Gamma_1; \dots; \Gamma_{p-1}; \Gamma_p - \{\gamma_{p,i}\}; \Gamma_{p+1} \cup \{\gamma_{p,i}\}; \Gamma_{p+2}; \dots; \Gamma_N),$$

$$\Gamma^{i'} = (\Gamma_1; \dots; \Gamma_{p-1}; \Gamma_p \cup \{\gamma_{p+1,i}\}; \Gamma_{p+1} - \{\gamma_{p+1,i}\}; \Gamma_{p+2}; \dots; \Gamma_N).$$

Similarly,

$$\rho^-(E_p(u)) : H_{\lambda - \alpha_p} \mapsto H_{\lambda},$$

$$\rho^-(E_p(u)) : [f] \mapsto \left[\sum_{i=1}^{\lambda_p} \frac{f(\mathbf{z}; \Gamma'^i; h)}{u - \gamma_{p,i}} \prod_{\substack{j=1 \\ j \neq i}}^{\lambda_p} \frac{1}{\gamma_{p,j} - \gamma_{p,i}} \prod_{k=1}^{\lambda_{p+1}} (\gamma_{p,i} - \gamma_{p+1,k} + h) \right],$$

$$\rho^-(F_p(u)) : H_{\lambda + \alpha_p} \mapsto H_{\lambda},$$

$$\rho^-(F_p(u)) : [f] \mapsto \left[\sum_{i=1}^{\lambda_{p+1}} \frac{f(\mathbf{z}; \Gamma^{i'}; h)}{u - \gamma_{p+1,i}} \prod_{\substack{j=1 \\ j \neq i}}^{\lambda_{p+1}} \frac{1}{\gamma_{p+1,i} - \gamma_{p+1,j}} \prod_{k=1}^{\lambda_p} (\gamma_{p,k} - \gamma_{p+1,i} + h) \right].$$

Proof. The statement follows from formulae (5.8)–(5.10) and (4.11), (4.12). \square

The $\tilde{Y}(\mathfrak{gl}_N)$ -module structures ρ^\pm on $\bigoplus_\lambda H_\lambda$ are the Yangian versions of representations of the quantum affine algebra $U_q(\widehat{\mathfrak{gl}_N})$ considered in [Vas1, Vas2]. The $U_q(\widehat{\mathfrak{gl}_N})$ analog of the ρ^+ structure on $\bigoplus_\lambda H_\lambda$ was considered in [GV]. The ρ^+ structure on $\bigoplus_\lambda H_\lambda$ was considered in [Va].

The ρ^- -structure appears to be more preferable, it was used in [RTVZ] to construct quantized conformal blocks in the tensor power $V^{\otimes n}$, see also Sections 6–8.

Corollary 5.11. *The $\tilde{Y}(\mathfrak{gl}_N)$ -module structures ρ^\pm on $\bigoplus_\lambda H_\lambda$ are contravariantly related through the Poincare pairing on \mathcal{F}_λ ,*

$$\int [f] \rho^+(X) [g] = \int [g] \rho^-(\varpi(X)) [f]$$

for any $X \in \tilde{Y}(\mathfrak{gl}_N)$ and any $f, g \in \bigoplus_\lambda H_\lambda$.

Proof. The statement follows from Proposition 5.2 and Lemma 4.2. \square

Corollary 5.12. *The $\tilde{Y}(\mathfrak{gl}_N)$ -module structure ρ^- on $\bigoplus_\lambda H_\lambda$ is contravariantly related to itself through the Poincare pairing on $T^*\mathcal{F}_\lambda$,*

$$\int [f] \rho^-(X) [g] = \int [g] \rho^-(\varpi(X)) [f]$$

for any $X \in \tilde{Y}(\mathfrak{gl}_N)$ and any $f, g \in \bigoplus_\lambda H_\lambda$.

Proof. The statement follows from relation (5.15), Proposition 5.3 and Lemma 4.2. \square

Corollary 5.13. *For any $X \in \tilde{\mathcal{B}}^q$ or $X \in \tilde{\mathcal{B}}^\infty$, and any $f, g \in \bigoplus_\lambda H_\lambda$, we have*

$$\int [f] \rho^+(X) [g] = \int [g] \rho^-(X) [f], \quad \int [f] \rho^-(X) [g] = \int [g] \rho^-(X) [f].$$

Proof. The statement follows from Corollaries 5.11, 5.12 and 4.3. \square

The $\tilde{Y}(\mathfrak{gl}_N)$ -module structures on $\bigoplus_\lambda H_\lambda$ descend to two $Y(\mathfrak{gl}_N)$ -module structures on the cohomology with constant coefficients

$$H(\mathbb{C}) := \bigoplus_{\lambda \in \mathbb{Z}_{\geq 0}^N, |\lambda|=n} H^*(T^*\mathcal{F}_\lambda; \mathbb{C}),$$

denoted by the same letters ρ^+ and ρ^- . The obtained $Y(\mathfrak{gl}_N)$ -modules are isomorphic to the irreducible $Y(\mathfrak{gl}_N)$ -modules $V^{(\pm)}$, and hence isomorphic among themselves, see Proposition 4.9.

Corollary 5.14. *The $Y(\mathfrak{gl}_N)$ -module $H(\mathbb{C})$ with the ρ^+ -structure is contravariantly dual to the $Y(\mathfrak{gl}_N)$ -module $H(\mathbb{C})$ with the ρ^- -structure with respect to the Poincare pairing on \mathcal{F}_λ ,*

$$\int [f] \rho^+(X) [g] = \int [g] \rho^-(\varpi(X)) [f]$$

for any $X \in \tilde{Y}(\mathfrak{gl}_N)$ and any $f, g \in \bigoplus_\lambda H_\lambda$.

Proof. The statement follows from Corollary 5.11. \square

Corollary 5.15. *The $Y(\mathfrak{gl}_N)$ -module $H(\mathbb{C})$ with the ρ^- -structure is self-dual with respect to the Poincare pairing on $T^*\mathcal{F}_\lambda$,*

$$\int [f] \rho^-(X) [g] = \int [g] \rho^-(\varpi(X)) [f]$$

for any $X \in \tilde{Y}(\mathfrak{gl}_N)$ and any $f, g \in \bigoplus_\lambda H_\lambda$.

Proof. The statement follows from Corollary 5.12. \square

5.5. Topological interpretation of the Yangian actions on cohomology. The $\tilde{Y}(\mathfrak{gl}_N)$ -actions ρ^\pm on $\bigoplus_\lambda H_\lambda$ have topological interpretations. Consider the partitions $\mu' = (\lambda_1, \dots, \lambda_p - 1, 1, \lambda_{p+1}, \dots, \lambda_N)$ and $\mu'' = (\lambda_1, \dots, \lambda_p, 1, \lambda_{p+1} - 1, \dots, \lambda_N)$. There are natural forgetful maps

$$(5.18) \quad \mathcal{F}_{\lambda - \alpha_p} \xleftarrow{\pi'_1} \mathcal{F}_{\mu'} \xrightarrow{\pi'_2} \mathcal{F}_\lambda, \quad \mathcal{F}_{\lambda + \alpha_p} \xleftarrow{\pi''_1} \mathcal{F}_{\mu''} \xrightarrow{\pi''_2} \mathcal{F}_\lambda.$$

The rank $\lambda_p - 1, 1, \lambda_{p+1}$ bundles over $\mathcal{F}_{\mu'}$ with fibers $F_p/F_{p-1}, F_{p+1}/F_p, F_{p+2}/F_{p+1}$ will be respectively denoted by A', B', C' . The rank $\lambda_p, 1, \lambda_{p+1} - 1$ bundles over $\mathcal{F}_{\mu''}$ with fibers $F_p/F_{p-1}, F_{p+1}/F_p, F_{p+2}/F_{p+1}$ will be respectively denoted by A'', B'', C'' .

For a T^n -equivariant bundle ξ , let $e(\xi)$ be its equivariant Euler class. We can make the extra \mathbb{C}^* (whose Chern root is h) act on any bundle by fiberwise action with weight $k \cdot h$ ($k \in \mathbb{Z}$). The equivariant Euler class with this extra action will be denoted by $e_{k \cdot h}(\xi)$. Note that B' and B'' are line bundles, hence their Euler class is their first Chern class.

Recall that a proper map $f : X \rightarrow Y$ induces the pullback $f^* : H^*(Y) \rightarrow H^*(X)$ and push-forward (also known as Gysin) $f_* : H^*(X) \rightarrow H^*(Y)$ maps on cohomology.

Theorem 5.16. *The operators $\rho^\pm(E_p(u)), \rho^\pm(F_p(u))$, are equal to the following topological operations*

$$\begin{aligned} \rho^+(E_p(u)) : x &\mapsto \pi'_{2*} \left(\pi'^*_1(x) \cdot \frac{e_{-h}(\text{Hom}(A', B'))}{u - e(B')} \right), \\ \rho^+(F_p(u)) : x &\mapsto \pi''_{2*} \left(\pi''^*_1(x) \cdot \frac{e_{-h}(\text{Hom}(B'', C''))}{u - e(B'')} \right), \\ \rho^-(E_p(u)) : x &\mapsto (-1)^{\lambda_p - \lambda_{p+1} + 1} \pi'_{2*} \left(\pi'^*_1(x) \cdot \frac{e_{-h}(\text{Hom}(B', C'))}{u - e(B')} \right), \\ \rho^-(F_p(u)) : x &\mapsto (-1)^{\lambda_p - \lambda_{p+1} + 1} \pi''_{2*} \left(\pi''^*_1(x) \cdot \frac{e_{-h}(\text{Hom}(A'', B''))}{u - e(B'')} \right). \end{aligned}$$

This theorem for the ρ^+ -structure was conjectured in [GV] and proved by M. Varagnolo in [Va]. Our proof below is different from the proof in [Va], where the author follows [Na] and checks that the right hand sides of formulae for ρ^+ satisfy the Yangian relations.

Proof. By Theorem 5.10, we already know that the ρ^\pm -structures on $\bigoplus_{\lambda} H_{\lambda}$ are Yangian module structures. Hence we only need to check that formulae of Theorem 5.10 are given by the right hand sides of formulae in Theorem 5.16.

The proof is a straightforward application of the equivariant localization formulation of push-forward maps. We omit the details, because they are completely analogous to the arguments in the Appendix of [RSTV]. Notice that the sign $(-1)^{\lambda_p - \lambda_{p+1} + 1}$ in the formulae for ρ^- comes from the fact that the functions $R(\mathbf{z}_I)$ here and $R(\mathbf{z}_{I_1} | \dots | \mathbf{z}_{I_N})$ in [RSTV] differ by the sign $(-1)^{\sum_{i < j} \lambda_i \lambda_j}$. \square

Remark. Observe that the Euler classes in the four expressions above are Euler classes of fiberwise tangent or cotangent bundles of some of the forgetful maps (5.18), following the convention of Section 5.1 that \mathbb{C}^* acts on these bundles by fiberwise multiplication.

6. BETHE ALGEBRAS $\tilde{\mathcal{B}}^q(\mathcal{V}_{\lambda}^+)$, $\tilde{\mathcal{B}}^q(\frac{1}{D}\mathcal{V}_{\lambda}^-)$, AND DISCRETE WRONSKIAN

The algebra H_{λ} , see (5.1), is a free polynomial algebra over \mathbb{C} with generators h and elementary symmetric functions $\sigma_i(\gamma_{p,1}, \dots, \gamma_{p,\lambda_p})$ for $p = 1, \dots, N$, $i = 1, \dots, \lambda_p$. Hence, H_{λ} is a free polynomial algebra over \mathbb{C} with generators h and the elements $f_{p,s}$, see (5.12), for $p = 1, \dots, N$, $s = 1, \dots, \lambda_p$. Then Theorems 5.4, 5.6 show that the Bethe algebras $\tilde{\mathcal{B}}^{\infty}(\mathcal{V}_{\lambda}^+)$ and $\tilde{\mathcal{B}}^{\infty}(\frac{1}{D}\mathcal{V}_{\lambda}^-)$ are free polynomial algebras over \mathbb{C} with generators h and the respective elements $C_{p,s}^{\pm}$, see (5.13), for $p = 1, \dots, N$, $s = 1, \dots, \lambda_p$.

In this section we give similar statements for the Bethe algebras $\tilde{\mathcal{B}}^q(\mathcal{V}_{\lambda}^+)$ and $\tilde{\mathcal{B}}^q(\frac{1}{D}\mathcal{V}_{\lambda}^-)$. We also formulate counterparts of Theorems 5.4, 5.6, see Theorems 6.3, 6.5 below.

In the rest of the paper we assume that q_1, \dots, q_N are distinct numbers. Set

$$(6.1) \quad W^q(u) = \det \left(q_i^{N-j} \prod_{k=1}^{\lambda_i} (u - \gamma_{i,k} + h(i-j)) \right)_{i,j=1}^N.$$

The function $W^q(u)$ is essentially a Casorati determinant (discrete Wronskian) of functions $g_i(u) = q_i^{u/h} \prod_{j=1}^{\lambda_i} (u - \gamma_{i,j} + h(i-1))$,

$$W^q(u) = \det \left(g_i(u - h(j-1)) \right)_{i,j=1}^N \prod_{i=1}^N q_i^{N-1-u/h}.$$

Define the algebra

$$(6.2) \quad \mathcal{H}_{\lambda}^q = \mathbb{C}[\mathbf{z}; \Gamma]^{S_n \times S_{\lambda_1} \times \dots \times S_{\lambda_N}} \otimes \mathbb{C}[h] \left/ \left\langle W^q(u) = \prod_{1 \leq i < j \leq N} (q_i - q_j) \prod_{a=1}^n (u - z_a) \right\rangle \right.$$

For example, if $N = n = 2$ and $\lambda = (1, 1)$, then the relations are

$$\gamma_{1,1} + \gamma_{2,1} = z_1 + z_2, \quad \gamma_{1,1} \gamma_{2,1} + \frac{q_2}{q_1 - q_2} h(\gamma_{1,1} - \gamma_{2,1} + h) = z_1 z_2.$$

It is easy to see that the subalgebra \mathcal{H}_{λ}^q does not change if all q_1, \dots, q_N are multiplied simultaneously by the same number. Notice that in the limit $q_{i+1}/q_i \rightarrow 0$ for all $i = 1, \dots, N-1$, the relations in \mathcal{H}_{λ}^q turn into the relations in H_{λ} , see (5.1).

Below we describe isomorphisms of the regular representation of \mathcal{H}_λ^q with the $\tilde{\mathcal{B}}^q(\mathcal{V}_\lambda^+)$ -module \mathcal{V}_λ^+ , as well as with the $\tilde{\mathcal{B}}^q(\frac{1}{D}\mathcal{V}_\lambda^-)$ -module $\frac{1}{D}\mathcal{V}_\lambda^-$.

Let x be a complex variable. Set

$$(6.3) \quad \widehat{W}^q(u, x) = \det \left(q_i^{N-j} \prod_{k=1}^{\lambda_i} (u - \gamma_{i,k} + h(i-j)) \right)_{i,j=0}^N,$$

where $q_0 = x$ and $\lambda_0 = 0$. Clearly, $\widehat{W}^q(u, x) = x^N W^q(u) + \dots + (-1)^N W^q(u + h)$. Define the elements $W_{p,s}^q \in \mathcal{H}_\lambda^q$ for $p = 1, \dots, N$, $s \in \mathbb{Z}_{>0}$, by the rule

$$(6.4) \quad \frac{\widehat{W}^q(u, x)}{W^q(u)} = \prod_{i=1}^N (x - q_i) + h \sum_{p=1}^N \sum_{s=1}^{\infty} (-1)^p W_{p,s}^q x^{N-p} u^{-s}.$$

Define also the elements $U_{i,s}^q \in \mathcal{H}_\lambda^q$ for $i = 1, \dots, N$, $s \in \mathbb{Z}_{>0}$ by the rule

$$(6.5) \quad U_{i,s}^q = \sum_{p=1}^N (-1)^{p-1} W_{p,s}^q q_i^{N-p-1} \prod_{\substack{j=1 \\ j \neq i}}^N \frac{1}{q_i - q_j},$$

so that

$$\frac{\widehat{W}^q(u, x)}{W^q(u)} \prod_{i=1}^N \frac{1}{x - q_i} = 1 - h \sum_{i=1}^N \sum_{s=1}^{\infty} \frac{q_i}{x - q_i} U_{i,s}^q u^{-s}.$$

Proposition 6.1. *The algebra \mathcal{H}_λ^q is a free polynomial algebra over \mathbb{C} with generators h and $U_{i,s}^q$ for $i = 1, \dots, N$, $s = 2, \dots, \lambda_i + 1$.*

Proof. Formulae (6.1) and (6.3)–(6.5) imply that $U_{i,1}^q = \lambda_i$ for all $i = 1, \dots, N$, and

$$(6.6) \quad U_{i,s}^q = (s-1) \sigma_{s-1}(\gamma_{i,1}, \dots, \gamma_{i,\lambda_i}) + U_{i,s}^{\prime q}, \quad s \geq 1,$$

where σ_{s-1} is the $(s-1)$ -th elementary symmetric function and $U_{i,s}^{\prime q}$ is a polynomial in h and $\sigma_r(\gamma_{j,1}, \dots, \gamma_{j,\lambda_j})$ for $r < s-1$ and $j = 1, \dots, N$. For instance,

$$(6.7) \quad U_{i,2}^q = \sigma_1(\gamma_{i,1}, \dots, \gamma_{i,\lambda_i}) - h i \lambda_i + \frac{h}{2} \lambda_i (\lambda_i + 1) - h \lambda_i \sum_{\substack{j=1 \\ j \neq i}}^N \frac{q_j (\lambda_j + 1)}{q_i - q_j}.$$

Since \mathcal{H}_λ^q is clearly a free polynomial algebra over \mathbb{C} with generators h and $\sigma_s(\gamma_{i,1}, \dots, \gamma_{i,\lambda_i})$ for $i = 1, \dots, N$, $s = 1, \dots, \lambda_i$, the statement follows. \square

Corollary 6.2. *The elements $W_{p,s}^q$ for $p = 1, \dots, N$, $s \geq 2$, together with $\mathbb{C}[h]$ generate the algebra \mathcal{H}_λ^q .* \square

Theorem 6.3 ([MTV5]). *The assignment $\mu_\lambda^{q+} : W_{p,s}^q \mapsto \phi^+(h^{s-1} B_{p,s}^q)$, $p = 1, \dots, N$, $s > 0$, extends uniquely to an isomorphism of $\mathbb{C}[z_1, \dots, z_n]^{S_n} \otimes \mathbb{C}[h]$ -algebras $\mu_\lambda^{q+} : \mathcal{H}_\lambda^q \rightarrow \tilde{\mathcal{B}}^q(\mathcal{V}_\lambda^+)$. The map*

$$(6.8) \quad \nu_\lambda^{q+} : \mathcal{H}_\lambda^q \rightarrow \mathcal{V}_\lambda^+, \quad [f] \mapsto \mu_\lambda^{q+}([f]) v_\lambda^+,$$

is an isomorphism of vector spaces identifying the $\tilde{\mathcal{B}}^q(\mathcal{V}_\lambda^+)$ -module \mathcal{V}_λ^+ and the regular representation of \mathcal{H}_λ^q .

Theorem 6.4 ([MTV5]). *The assignment $\mu_\lambda^{q=}: W_{p,s}^q \mapsto \phi^+(h^{s-1}B_{p,s}^q)$, $p = 1, \dots, N$, $s > 0$, extends uniquely to an isomorphism of $\mathbb{C}[z_1, \dots, z_n]^{S_n} \otimes \mathbb{C}[h]$ -algebras $\mu_\lambda^{q=}: \mathcal{H}_\lambda^q \rightarrow \tilde{\mathcal{B}}^q(\frac{1}{D}\mathcal{V}_\lambda^-)$. The map*

$$(6.9) \quad \nu_\lambda^{q=}: \mathcal{H}_\lambda^q \rightarrow \frac{1}{D}\mathcal{V}_\lambda^-, \quad [f] \mapsto \mu_\lambda^{q=}([f]) \nu_\lambda^-,$$

is an isomorphism of vector spaces identifying the $\tilde{\mathcal{B}}^q(\frac{1}{D}\mathcal{V}_\lambda^-)$ -module $\frac{1}{D}\mathcal{V}_\lambda^-$ and the regular representation of \mathcal{H}_λ^q .

Theorem 6.5 ([MTV5]). *The assignment $\mu_\lambda^{q-}: W_{p,s}^q \mapsto \phi^-(h^{s-1}B_{p,s}^q)$, $p = 1, \dots, N$, $s > 0$, extends uniquely to an isomorphism of $\mathbb{C}[z_1, \dots, z_n]^{S_n} \otimes \mathbb{C}[h]$ -algebras $\mu_\lambda^{q-}: \mathcal{H}_\lambda^q \rightarrow \tilde{\mathcal{B}}^q(\frac{1}{D}\mathcal{V}_\lambda^-)$. The map*

$$(6.10) \quad \nu_\lambda^{q-}: \mathcal{H}_\lambda^q \rightarrow \frac{1}{D}\mathcal{V}_\lambda^-, \quad [f] \mapsto \mu_\lambda^{q-}([f]) \nu_\lambda^-,$$

is an isomorphism of vector spaces identifying the $\tilde{\mathcal{B}}^q(\frac{1}{D}\mathcal{V}_\lambda^-)$ -module $\frac{1}{D}\mathcal{V}_\lambda^-$ and the regular representation of \mathcal{H}_λ^q .

Theorems 6.3–6.5 are instances of the discrete geometric Langlands correspondence, see the corresponding versions of the geometric Langlands correspondence in [MTV2, MTV3], cf. also [RSTV].

Theorems 6.4 and 6.5 are equivalent by Lemma 4.5. In particular,

$$(6.11) \quad \tilde{\Pi} \mu_\lambda^{q=}(f) = \mu_\lambda^{q-}(f) \tilde{\Pi}, \quad \tilde{\Pi} \nu_\lambda^{q=} = \nu_\lambda^{q-}$$

where $\tilde{\Pi}$ is given by (4.4) and $f \in \mathcal{H}_\lambda^q$.

Corollary 6.6. *The $\tilde{\mathcal{B}}^q$ -modules \mathcal{V}^+ , $\frac{1}{D}\mathcal{V}^=$, and $\frac{1}{D}\mathcal{V}^-$ are isomorphic.*

Proof. The isomorphisms restricted to weight subspaces are $\nu_\lambda^{q=}(\nu_\lambda^{q+})^{-1}: \mathcal{V}_\lambda^+ \rightarrow \frac{1}{D}\mathcal{V}_\lambda^=$ and $\nu_\lambda^{q-}(\nu_\lambda^{q+})^{-1}: \mathcal{V}_\lambda^+ \rightarrow \frac{1}{D}\mathcal{V}_\lambda^-$. \square

Corollary 6.7. *The subalgebras $\tilde{\mathcal{B}}^q(\mathcal{V}_\lambda^+) \subset \text{End}(\mathcal{V}_\lambda^+)$, $\tilde{\mathcal{B}}^q(\frac{1}{D}\mathcal{V}_\lambda^=) \subset \text{End}(\frac{1}{D}\mathcal{V}_\lambda^=)$ and $\tilde{\mathcal{B}}^q(\frac{1}{D}\mathcal{V}_\lambda^-) \subset \text{End}(\frac{1}{D}\mathcal{V}_\lambda^-)$ are maximal commutative subalgebras.* \square

Corollary 6.8. *The Bethe algebra $\tilde{\mathcal{B}}^q(\mathcal{V}_\lambda^+)$ is a free polynomial algebra over \mathbb{C} with generators h and $\phi^+(S_{p,s}^q)$ for $p = 1, \dots, N$, $s = 1, \dots, \lambda_p$, where the elements $S_{p,s}^q \in \tilde{\mathcal{B}}^q$ are defined by (3.9). The Bethe algebra $\tilde{\mathcal{B}}^q(\frac{1}{D}\mathcal{V}_\lambda^=)$ is a free polynomial algebra over \mathbb{C} with generators h and $\phi^+(S_{p,s}^q)$ for $p = 1, \dots, N$, $s = 1, \dots, \lambda_p$. The Bethe algebra $\tilde{\mathcal{B}}^q(\frac{1}{D}\mathcal{V}_\lambda^-)$ is a free polynomial algebra over \mathbb{C} with generators h and $\phi^-(S_{p,s}^q)$ for $p = 1, \dots, N$, $s = 1, \dots, \lambda_p$.*

Proof. Since $\mu_\lambda^{q+}(U_{i,s}) = \mu_\lambda^{q=}(U_{i,s}) = \phi^+(S_{i,s}^q)$ and $\mu_\lambda^{q-}(U_{i,s}) = \phi^-(S_{i,s}^q)$, see (6.5), (3.9), the statement follows from Theorems 6.3–6.5, and Proposition 6.1. \square

Lemma 6.9. *For any $f \in \mathcal{H}_\lambda^q$, and any $g_1 \in \mathcal{V}_\lambda^+$, $g_2 \in \frac{1}{D}\mathcal{V}_\lambda^-$, $g_3 \in \frac{1}{D}\mathcal{V}_\lambda^-$, we have*

$$\mathcal{S}(\mu_\lambda^{q+}(f)g_1, g_3) = \mathcal{S}(g_1, \mu_\lambda^{q-}(f)g_3), \quad \mathcal{S}(\mu_\lambda^{q-}(f)g_2, g_3) = \mathcal{S}(g_2, \mu_\lambda^{q-}(f)g_3).$$

Proof. The statement follows from the definition of the maps μ_λ^\pm and Corollary 4.3. \square

Recall that $\tilde{Y}(\mathfrak{gl}_N)$ contains $U(\mathfrak{gl}_N)$ as a subalgebra. Denote by $\tilde{Y}(\mathfrak{gl}_N)^\mathfrak{h}$ the subalgebra of $\tilde{Y}(\mathfrak{gl}_N)$ commuting with $U(\mathfrak{h})$, where $\mathfrak{h} \subset \mathfrak{gl}_N$ in the Lie subalgebra generated by $e_{i,i}$, $i = 1, \dots, N$. The isomorphisms $\nu_\lambda^{q\pm}$ induce homomorphisms

$$(6.12) \quad \rho_\lambda^{q\pm} : \tilde{Y}(\mathfrak{gl}_N)^\mathfrak{h} \rightarrow \text{End}(\mathcal{H}_\lambda^q), \quad \rho_\lambda^{q\pm} : X \mapsto (\nu_\lambda^{q\pm})^{-1} \phi^\pm(X) \nu_\lambda^{q\pm}.$$

By relations (6.11) and (4.5), we also have $\rho^{q-}(X) = (\nu_\lambda^{q-})^{-1} \phi^+(X) \nu_\lambda^{q-}$.

Lemma 6.10. *For $i = 1, \dots, N$, we have*

$$\begin{aligned} \rho_\lambda^{q\pm}(X_{\lambda,i}^{q-}) : [f] \mapsto & \left[f(z; \Gamma; h) \left(\sum_{k=1}^{\lambda_i} \gamma_{i,k} + h \sum_{j=1}^{i-1} \frac{q_i}{q_i - q_j} \min(0, \lambda_j - \lambda_i) + \right. \right. \\ & \left. \left. + h \sum_{j=i+1}^N \frac{q_j}{q_i - q_j} \min(0, \lambda_j - \lambda_i) \right) \right]. \end{aligned}$$

Proof. The statement follows from formulae (6.7), (3.13), (3.17) because $\rho_\lambda^{q\pm}(S_{i,2}^q)$ acts as multiplication by the element $U_{i,2}^q$. \square

Define pairings

$$(6.13) \quad (,) : \mathcal{H}_\lambda^q \otimes \mathcal{H}_\lambda^q \rightarrow \mathbb{C}[z_1, \dots, z_n]^{S_n} \otimes \mathbb{C}[h], \quad (f, g) = \mathcal{S}(\nu_\lambda^{q+}(f), \nu_\lambda^{q-}(g)),$$

and

$$(6.14) \quad \langle , \rangle : \mathcal{H}_\lambda^q \otimes \mathcal{H}_\lambda^q \rightarrow Z^{-1} \mathbb{C}[z_1, \dots, z_n]^{S_n} \otimes \mathbb{C}[h], \quad \langle f, g \rangle = \mathcal{S}(\nu_\lambda^{q-}(f), \nu_\lambda^{q-}(g)).$$

Here $Z = \prod_{i \neq j} (z_i - z_j + h)$, cf. (2.14).

Lemma 6.11. *Pairings (6.13) and (6.14) are symmetric and invariant,*

$$\begin{aligned} (f_1, f_2) &= (f_2, f_1), & (f_1 f_2, f_3) &= (f_2, f_1 f_3), \\ \langle f_1, f_2 \rangle &= \langle f_2, f_1 \rangle, & \langle f_1 f_2, f_3 \rangle &= \langle f_2, f_1 f_3 \rangle, \end{aligned}$$

for any $f_1, f_2, f_3 \in \mathcal{H}_\lambda^q$.

Proof. Theorems 6.3–6.5 and Corollary 4.3 yield $(f_1, f_2) = (1, f_1 f_2)$ and $\langle f_1, f_2 \rangle = \langle 1, f_1 f_2 \rangle$, which proves the statement. \square

Example. Let $N = n = 2$, $\lambda = (1, 1)$, so that $V_\lambda = \mathbb{C}v_{(1,2)} \oplus \mathbb{C}v_{(2,1)}$. Then

$$\mu_\lambda^{q+} : [1] \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mu_\lambda^{q+} : [\gamma_{1,1}] \mapsto \begin{pmatrix} z_1 & h \\ 0 & z_2 \end{pmatrix} + \frac{q_2}{q_1 - q_2} \begin{pmatrix} h & h \\ h & h \end{pmatrix},$$

$$\begin{aligned}
\nu_{\lambda}^{q+} : [1] &\mapsto v_{(1,2)} + v_{(2,1)}, \\
\nu_{\lambda}^{q+} : [\gamma_{1,1}] &\mapsto \left(z_1 + h \frac{q_1 + q_2}{q_1 - q_2}\right) v_{(1,2)} + \left(z_2 + \frac{2hq_2}{q_1 - q_2}\right) v_{(2,1)}, \\
\mu_{\lambda}^{q=} : [1] &\mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mu_{\lambda}^{q=} : [\gamma_{1,1}] \mapsto \begin{pmatrix} z_1 & h \\ 0 & z_2 \end{pmatrix} + \frac{q_2}{q_1 - q_2} \begin{pmatrix} h & h \\ h & h \end{pmatrix}, \\
\nu_{\lambda}^{q=} : [1] &\mapsto \frac{v_{(1,2)} - v_{(2,1)}}{z_1 - z_2 - h}, \quad \nu_{\lambda}^{q=} : [\gamma_{1,1}] \mapsto \frac{(z_1 - h)v_{(1,2)} - z_2 v_{(2,1)}}{z_1 - z_2 - h}, \\
\mu_{\lambda}^{q-} : [1] &\mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mu_{\lambda}^{q-} : [\gamma_{1,1}] \mapsto \begin{pmatrix} z_1 & 0 \\ h & z_2 \end{pmatrix} + \frac{q_2}{q_1 - q_2} \begin{pmatrix} h & h \\ h & h \end{pmatrix}, \\
\nu_{\lambda}^{q-} : [1] &\mapsto \frac{v_{(1,2)} - v_{(2,1)}}{z_1 - z_2 + h}, \quad \nu_{\lambda}^{q-} : [\gamma_{1,1}] \mapsto \frac{z_1 v_{(1,2)} - (z_2 - h)v_{(2,1)}}{z_1 - z_2 + h}, \\
(1, 1) &= 0, \quad (1, \gamma_{1,1}) = 1, \quad (\gamma_{1,1}, \gamma_{1,1}) = z_1 + z_2 + \frac{2hq_2}{q_1 - q_2}. \\
\langle 1, 1 \rangle &= \frac{2}{(z_1 - z_2 + h)(z_1 - z_2 - h)}, \quad \langle 1, \gamma_{1,1} \rangle = \frac{z_1 + z_2 - h}{(z_1 - z_2 + h)(z_1 - z_2 - h)}, \\
\langle \gamma_{1,1}, \gamma_{1,1} \rangle &= \frac{z_1^2 + z_2^2 - h(z_1 + z_2)}{(z_1 - z_2 + h)(z_1 - z_2 - h)}.
\end{aligned}$$

Notice that up to a simple change of basis, the form $\langle \cdot, \cdot \rangle$ in this example coincides with the bilinear form $\langle \cdot, \cdot \rangle$ in [BGr, Section 3.2], namely, with the form associated with the equivariant quantum cohomology of the cotangent bundle $T^*\mathbb{P}^1$ of the projective line \mathbb{P}^1 . That form was used in [BMO], see also [BGh].

7. QUANTUM MULTIPLICATION ON H_{λ}

Recall that q_1, \dots, q_N are distinct numbers. Let $\beta_{\lambda}^{\pm} : H_{\lambda} \rightarrow \mathcal{H}_{\lambda}^q$ be the following isomorphisms of vector spaces

$$\beta_{\lambda}^{+} = (\nu_{\lambda}^{q+})^{-1} \nu_{\lambda}^{+}, \quad \beta_{\lambda}^{-} = (\nu_{\lambda}^{q-})^{-1} \nu_{\lambda}^{-}.$$

We also have $\beta_{\lambda}^{-} = (\nu_{\lambda}^{q-})^{-1} \nu_{\lambda}^{-}$, because $\nu_{\lambda}^{-} = \tilde{\Pi} \nu_{\lambda}^{-}$ and $\nu_{\lambda}^{q-} = \tilde{\Pi} \nu_{\lambda}^{q-}$, where $\tilde{\Pi}$ is given by (4.4). Notice that $\beta_{\lambda}^{\pm}(1) = 1 \in \mathcal{H}_{\lambda}^q$.

Define new multiplications \star and \bullet on H_{λ} by the rule

$$(7.1) \quad \beta_{\lambda}^{+}(f \star g) = \beta_{\lambda}^{+}(f) \beta_{\lambda}^{+}(g), \quad \beta_{\lambda}^{-}(f \bullet g) = \beta_{\lambda}^{-}(f) \beta_{\lambda}^{-}(g)$$

for any $f, g \in H_{\lambda}$. The defined multiplications are associative and commutative. The products $f \star g$ and $f \bullet g$ tend to the ordinary product fg as $q_{i+1}/q_i \rightarrow 0$ for every $i = 1, \dots, N-1$,

Denote $f \star$ and $f \bullet$ the corresponding operators on H_{λ} , $f \star : g \mapsto f \star g$, $f \bullet : g \mapsto f \bullet g$.

Recall the $\tilde{Y}(\mathfrak{gl}_N)$ -actions ρ^\pm on $\bigoplus_\lambda H_\lambda$, defined in Section 5.4. Given $X \in \tilde{Y}(\mathfrak{gl}_N)^\mathfrak{h}$, denote by $\rho_\lambda^\pm(X) \in \text{End}(H_\lambda)$ the restriction of $\rho^\pm(X)$ to H_λ .

Lemma 7.1. *Let $f \in H_\lambda$ and $X \in \tilde{\mathcal{B}}^q$. If $\nu_\lambda^+(f) = \phi^+(X) v_\lambda^+$, then $f \star = \rho_\lambda^+(X)$. Similarly, if $\nu_\lambda^-(f) = \phi^+(X) v_\lambda^-$ or $\nu_\lambda^-(f) = \phi^-(X) v_\lambda^-$, then $f \bullet = \rho_\lambda^-(X)$.*

Proof. We will prove the statement for the map ν_λ^- . Other statements are proved similarly. Since $\nu_\lambda^-(g) = \nu_\lambda^{q-}(\beta_\lambda^-(g))$ for any $g \in H_\lambda$, we have

$$\begin{aligned} \nu_\lambda^-(f \bullet g) &= \nu_\lambda^-(g \bullet f) = \mu_\lambda^{q-}(\beta_\lambda^-(g)) \nu_\lambda^-(f) = \mu_\lambda^{q-}(\beta_\lambda^-(g)) \phi^-(X) v_\lambda^- = \\ &= \phi^-(X) \mu_\lambda^{q-}(\beta_\lambda^-(g)) v_\lambda^- = \phi^-(X) \nu_\lambda^-(g) = \nu_\lambda^-(\rho^-(X) g), \end{aligned}$$

which proves the claim. \square

Denote by $\tilde{\mathcal{B}}_\lambda^{q\pm}$ the images in $\text{End}(H_\lambda)$ of the Bethe subalgebra $\tilde{\mathcal{B}}^q \subset \tilde{Y}(\mathfrak{gl}_N)$ under the homomorphisms ρ^\pm , respectively. The algebra $\tilde{\mathcal{B}}_\lambda^{q+}$ is isomorphic to $\tilde{\mathcal{B}}^q(\mathcal{V}_\lambda^+)$, and the algebra $\tilde{\mathcal{B}}_\lambda^{q-}$ is isomorphic to $\tilde{\mathcal{B}}^q(\frac{1}{D}\mathcal{V}_\lambda^-)$ and to $\tilde{\mathcal{B}}^q(\frac{1}{D}\mathcal{V}_\lambda^-)$.

Corollary 7.2. *For any $f \in H_\lambda$, the operator $f \star$ is the unique element of $\tilde{\mathcal{B}}_\lambda^{q+}$ which sends $1 \in H_\lambda$ to f , and the operator $f \bullet$ is the unique element of $\tilde{\mathcal{B}}_\lambda^{q-}$ which sends $1 \in H_\lambda$ to f .* \square

Corollary 7.3. *For every $i = 1, \dots, N$, we have*

$$(7.2) \quad (\gamma_{i,1} + \dots + \gamma_{i,\lambda_i}) \star = \rho_\lambda^+(X_i^{q+}), \quad (\gamma_{i,1} + \dots + \gamma_{i,\lambda_i}) \bullet = \rho_\lambda^-(X_{\lambda,i}^{q-}),$$

where $\gamma_{i,j}$ are the Chern roots, see (5.1), and $X_i^{q+}, X_{\lambda,i}^{q-} \in \tilde{\mathcal{B}}^q$ are given by (3.16), (3.17).

Proof. The statement follows from Lemmas 7.1 and 4.18. \square

Proposition 7.4. *The map $\chi_\lambda^+ : f \mapsto f \star$ is an isomorphism $H_\lambda \rightarrow \tilde{\mathcal{B}}_\lambda^{q+}$ of vector spaces. The map $\beta_\lambda^+(\chi_\lambda^+)^{-1} : \tilde{\mathcal{B}}_\lambda^{q+} \rightarrow \mathcal{H}_\lambda^q$ is an isomorphism of $\mathbb{C}[z_1, \dots, z_n]^{S_n} \otimes \mathbb{C}[h]$ -algebras. The maps $\beta_\lambda^+(\chi_\lambda^+)^{-1}$ and β_λ^+ identify the $\tilde{\mathcal{B}}_\lambda^{q+}$ -module H_λ with the regular representation of the algebra \mathcal{H}_λ^q .* \square

Proposition 7.5. *The map $\chi_\lambda^- : f \mapsto f \bullet$ is an isomorphism $H_\lambda \rightarrow \tilde{\mathcal{B}}_\lambda^{q-}$ of vector spaces. The map $\beta_\lambda^-(\chi_\lambda^-)^{-1} : \tilde{\mathcal{B}}_\lambda^{q-} \rightarrow \mathcal{H}_\lambda^q$ is an isomorphism of $\mathbb{C}[z_1, \dots, z_n]^{S_n} \otimes \mathbb{C}[h]$ -algebras. The maps $\beta_\lambda^-(\chi_\lambda^-)^{-1}$ and β_λ^- identify the $\tilde{\mathcal{B}}_\lambda^{q-}$ -module H_λ with the regular representation of the algebra \mathcal{H}_λ^q .* \square

Consider q_1, \dots, q_n as variables. For a nonzero complex number κ , define the connections ∇^\star and ∇_λ^\bullet

$$(7.3) \quad \nabla_i^\star = \kappa q_i \frac{\partial}{\partial q_i} - (\gamma_{i,1} + \dots + \gamma_{i,\lambda_i}) \star, \quad \nabla_{\lambda,i}^\bullet = \kappa q_i \frac{\partial}{\partial q_i} - (\gamma_{i,1} + \dots + \gamma_{i,\lambda_i}) \bullet,$$

$i = 1, \dots, n$.

Proposition 7.6. *The connections ∇^\star and ∇_λ^\bullet are flat for any κ .*

Proof. The statement follows from Corollary 7.3 and Lemma 3.5. \square

Lemma 7.7. *For any $f, g \in H_\lambda$, we have*

$$\int f g = \int f \bullet g = \langle \beta_\lambda^-(f), \beta_\lambda^-(g) \rangle,$$

where the pairing $\langle \cdot, \cdot \rangle$ on \mathcal{H}_λ^q is defined by (6.14).

Proof. Propositions 5.2, 5.3 and Lemma 6.11 imply

$$\begin{aligned} \int f g &= \mathcal{S}(\nu_\lambda^-(f), \nu_\lambda^-(g)) = \mathcal{S}(\nu_\lambda^{q^-}(\beta_\lambda^-(f)), \nu_\lambda^{q^-}(\beta_\lambda^-(g))) = \langle \beta_\lambda^-(f), \beta_\lambda^-(g) \rangle = \\ &= \langle 1, \beta_\lambda^-(f) \beta_\lambda^-(g) \rangle = \mathcal{S}(\nu_\lambda^-, \nu_\lambda^{q^-}(\beta_\lambda^-(f) \beta_\lambda^-(g))) = \mathcal{S}(\nu_\lambda^-, \nu_\lambda^-(f \bullet g)) = \int f \bullet g. \end{aligned} \quad \square$$

Corollary 7.8. *For any $f_1, f_2, f_3 \in H_\lambda$, we have*

$$\int (f_1 \bullet f_2) f_3 = \int f_1 \bullet f_2 \bullet f_3 = \int f_2 (f_1 \bullet f_3). \quad \square$$

Corollary 7.9. *The operators $f \bullet$ are symmetric with respect to pairing (5.7).*

In [MO], the quantum multiplication on $H_{T^n \times \mathbb{C}^*}^*(T^*\mathcal{F}_\lambda)$ is described. The quantum multiplication depends on parameters $q^{\alpha_1}, \dots, q^{\alpha_{N-1}}$ associated with simple roots $\alpha_1, \dots, \alpha_{N-1}$ of the Lie algebra \mathfrak{sl}_N .

We identify the notation of [MO] with the notation of this paper as follows:

$$(7.4) \quad h \mapsto -h, \quad q^{\alpha_i} \mapsto q_{i+1}/q_i, \quad i = 1, \dots, N-1.$$

and z_1, \dots, z_n are the same as in [MO].

Theorem 7.10. *Under the identification of notation (7.4), the space H_λ with the multiplication \bullet and pairing (5.7) is the equivariant quantum cohomology algebra $QH_{GL_n \times \mathbb{C}^*}(T^*\mathcal{F}_\lambda; \mathbb{C})$ of the cotangent bundle $T^*\mathcal{F}_\lambda$ of a partial flag variety \mathcal{F}_λ , described in [MO].*

Proof. Assume throughout the proof the identification of notation (7.4). By [RTV, Corollary 6.4], the Yangian structure on the equivariant cohomology defined in [MO] coincides with the Yangian structure ρ^- defined in Section 5.4. By [RTV, Corollary 7.6], the action of the dynamical Hamiltonians $X_1^{q^-}, \dots, X_N^{q^-}$ on the equivariant cohomology coincides with the quantum multiplication by the first Chern classes defined in [MO]. Assume that z_1, \dots, z_n are fixed, that is, we consider the corresponding quotients of the Bethe algebra and the algebra of quantum multiplication. Suppose that z_1, \dots, z_n are generic. Then by formula (5.17), the dynamical Hamiltonians $X_1^\infty, \dots, X_N^\infty$ have simple spectrum. Thus for generic q_1, \dots, q_N , the dynamical Hamiltonians $X_1^{q^-}, \dots, X_N^{q^-}$ have simple spectrum too, and hence generate the whole Bethe algebra (as well as the whole algebra of quantum multiplication). Both the Bethe algebra and the algebra of quantum multiplication continuously depend on parameters q_1, \dots, q_N and z_1, \dots, z_n as long as q_1, \dots, q_N are distinct. Hence, both algebras coincide for all distinct q_1, \dots, q_N and all fixed z_1, \dots, z_n . \square

Below in Section 8.4, we give a direct proof of the special case $N = n$ and $\lambda = (1, \dots, 1)$ of Theorem 7.10. The proof is based on formulae from [BMO].

Observe that under Theorem 7.10, the connection ∇^\bullet for $\kappa = -1$ is the quantum connection on H_λ .

Corollary 7.11. *Under the identification of notation (7.4), the equivariant quantum cohomology algebra $QH_{GL_n \times \mathbb{C}^*}(T^*\mathcal{F}_\lambda; \mathbb{C})$ of the cotangent bundle $T^*\mathcal{F}_\lambda$ of a partial flag variety \mathcal{F}_λ is isomorphic to the algebra \mathcal{H}_λ^q .*

This corollary gives a description of the algebra $QH_{GL_n \times \mathbb{C}^*}(T^*\mathcal{F}_\lambda; \mathbb{C})$ by generators and relations, see formula (6.2). Theorem 7.13 below implies that in the limit $h \rightarrow \infty$ such that all

$$r_i = h^{\lambda_i + \lambda_{i+1}} q_{i+1} / q_i, \quad i = 1, \dots, N-1,$$

are fixed, this description becomes the known description of the equivariant quantum cohomology algebra $QH_{GL_n}(\mathcal{F}_\lambda; \mathbb{C})$ of the partial flag variety \mathcal{F}_λ , see for instance [AS, formula (2.22)].

Introduce the $N \times N$ matrix $M(u)$ with entries $M_{i,j} = 0$ for $|i - j| > 1$,

$$M_{i,i}(u) = \prod_{k=1}^{\lambda_i} (u - \gamma_{i,k}), \quad i = 1, \dots, N,$$

$$M_{i,i+1}(u) = (-1)^{\lambda_i}, \quad M_{i+1,i}(u) = r_i, \quad i = 1, \dots, N-1.$$

Lemma 7.12. *Let A be the matrix used in [AS, formula (2.22)]. Match the notation of [AS] and this paper as follows:*

$$k \mapsto N, \quad n_i \mapsto \lambda_i, \quad x_j^{(i)} \mapsto \sigma_j(\gamma_{i,1}, \dots, \gamma_{i,\lambda_i}), \quad q_i \mapsto r_i,$$

where σ_j is the j -th elementary symmetric function. Then $\det(u - A) = \det M(u)$.

Proof. The claim follows by induction on N by expanding $\det(u - A)$ and $\det M(u)$ with respect to the last column. \square

Theorem 7.13. *Assume that $\lambda_i > 0$ for all $i = 1, \dots, N$, and $h \rightarrow \infty$ such that r_1, \dots, r_{N-1} are fixed. Then*

$$W^q(u) \prod_{1 \leq i < j \leq N} (q_i - q_j)^{-1} \rightarrow \det M(u).$$

Proof. Let $p_i = \prod_{k=1}^{i-1} h^{\lambda_k} (q_i / q_{i+1})^{N-i+1}$, $i = 1, \dots, N$. Then under the assumption of the theorem,

$$\prod_{1 \leq i < j \leq N} (q_i - q_j) = q_1^{N-1} q_2^{N-2} \dots q_{N-1} (1 + o(1)),$$

and

$$q_i^{N-j} \prod_{k=1}^{\lambda_i} (u - \gamma_{i,k} + h(i - j)) = q_j^{N-j} \frac{p_j}{p_i} (M_{i,j}(u) + o(1)),$$

which implies the claim. \square

8. SPECIAL CASE $N = n$, $\lambda = (1, \dots, 1)$, AND HECKE ALGEBRA

In this section we consider the special case when \mathcal{F}_λ is the variety of full flags in \mathbb{C}^n .

8.1. Hecke algebra. Consider the algebra \mathfrak{H}_n generated by the central element c , pairwise commuting elements y_1, \dots, y_n , and elements of the symmetric group S_n , subject to relations

$$(8.1) \quad \begin{aligned} s_i y_i - y_{i+1} s_i &= c, & i &= 1, \dots, n-1, \\ s_i y_j - y_j s_i &= 0, & j &\neq i, i+1, \end{aligned}$$

where s_i is the transposition of i and $i+1$. For every nonzero complex number t , the quotient algebra $\mathfrak{H}_n/(c=t)$ is the degenerate affine Hecke algebra corresponding to the parameter t .

For $i = 1, \dots, n$, set

$$(8.2) \quad Y_i^q = y_i + c \sum_{j=1}^{i-1} \frac{q_i}{q_i - q_j} s_{i,j} + c \sum_{j=i+1}^n \frac{q_j}{q_i - q_j} s_{i,j}.$$

where $s_{i,j} \in \mathfrak{H}_n$ is the transposition of i and j . It is known that for any nonzero complex number p , the formal differential operators

$$(8.3) \quad \nabla_i^{\mathfrak{H}} = \kappa q_i \frac{\partial}{\partial q_i} - Y_i^q, \quad i = 1, \dots, N,$$

pairwise commute and, hence, define a flat connection for any \mathfrak{H}_n -module. This connection is called the *affine KZ connection*, see [C, formula (1.1.41)].

8.2. The case $N = n$ and $\lambda = (1, \dots, 1)$. Recall that q_1, \dots, q_N are distinct numbers.

Theorem 8.1. *Let $N = n$ and $\lambda = (1, \dots, 1)$. Then the operators $\phi^+(X_i^q)$, $i = 1, \dots, N$, and $\mathbb{C}[z_1, \dots, z_n]^{S_n} \otimes \mathbb{C}[h]$ generate the Bethe algebra $\tilde{\mathcal{B}}^q(\mathcal{V}_\lambda^+)$. Similarly, the operators $\phi^+(X_i^q)$, $i = 1, \dots, N$, and $\mathbb{C}[z_1, \dots, z_n]^{S_n} \otimes \mathbb{C}[h]$ generate the Bethe algebra $\tilde{\mathcal{B}}^q(\frac{1}{D}\mathcal{V}_\lambda^-)$, and the operators $\phi^-(X_i^q)$, $i = 1, \dots, N$, and $\mathbb{C}[z_1, \dots, z_n]^{S_n} \otimes \mathbb{C}[h]$ generate the Bethe algebra $\tilde{\mathcal{B}}^q(\frac{1}{D}\mathcal{V}_\lambda^-)$.*

Proof. The statement follows from formula (3.13) and Corollary 6.8. □

For any $\sigma \in S_n$ define $w_\sigma \in \text{End}(V)$ by the rule

$$w_\sigma(v_{i_1} \otimes \dots \otimes v_{i_n}) = v_{\sigma(i_1)} \otimes \dots \otimes v_{\sigma(i_n)}$$

for any sequence i_1, \dots, i_n . The operators w_σ define the action of S_n on V . We will write $w_{(i,j)} = w_\sigma$ for σ being the transposition of i and j .

Recall the elements $G_{i,j} = e_{i,j} e_{j,i} - e_{i,i}$.

Lemma 8.2. *Let $N = n$ and $\lambda = (1, \dots, 1)$. For any $i \neq j$, the element $G_{i,j}$ acts on V_λ as $w_{(i,j)}$.* □

Proposition 8.3. *Let $N = n$ and $\lambda = (1, \dots, 1)$. The assignments $y_i \mapsto \phi^+(X_i^\infty)$, $c \mapsto h$, $s_j \mapsto G_{(j,j+1)}$, $i = 1, \dots, n$, $j = 1, \dots, n-1$, define a representation of \mathfrak{H}_n on \mathcal{V}_λ^+ . Similarly, the assignments $y_i \mapsto \phi^-(X_i^\infty)$, $c \mapsto -h$, $s_j \mapsto -G_{(j,j+1)}$, $i = 1, \dots, n$, $j = 1, \dots, n-1$, define a representation of \mathfrak{H}_n on $\frac{1}{D}\mathcal{V}_\lambda^-$.*

Proof. The verification is straightforward using formulae (4.13), (8.1), and Lemma 8.2. \square

Denote the obtained representations of \mathfrak{H}_n on \mathcal{V}_λ^+ and $\frac{1}{D}\mathcal{V}_\lambda^-$ respectively by ψ^+ and ψ^- . By Proposition 8.3 and formulae (3.15), (8.2), we have

$$(8.4) \quad \phi^\pm(X_i^q) = \psi^\pm(Y_i^q), \quad i = 1, \dots, n.$$

So for $N = n$ and $\lambda = (1, \dots, 1)$, the trigonometric dynamical connection (3.18) defined on \mathcal{V}_λ^+ through ϕ^+ coincides with the affine KZ connection (8.3) defined on \mathcal{V}_λ^+ through ψ^+ , and similarly, the trigonometric dynamical connection (3.18) defined on $\frac{1}{D}\mathcal{V}_\lambda^-$ through ϕ^- coincides with the affine KZ connection (8.3) defined on $\frac{1}{D}\mathcal{V}_\lambda^-$ through ψ^- .

Lemma 8.4. *Let $N = n$ and $\lambda = (1, \dots, 1)$. For the vectors $v_\lambda^+ \in \mathcal{V}_\lambda^+$ and $v_\lambda^- \in \frac{1}{D}\mathcal{V}_\lambda^-$, we have*

$$(8.5) \quad v_\lambda^+ = \sum_{\sigma \in S_n} v_{\sigma(1)} \otimes \dots \otimes v_{\sigma(n)}, \quad v_\lambda^- = \frac{1}{D} \sum_{\sigma \in S_n} (-1)^\sigma v_{\sigma(1)} \otimes \dots \otimes v_{\sigma(n)}.$$

Thus $w_{(i,j)} v_\lambda^\pm = \pm v_\lambda^\pm$ for any $i \neq j$. \square

For $N = n$ and $\lambda = (1, \dots, 1)$, denote the Chern roots $\gamma_{1,1}, \dots, \gamma_{n,1}$ of the corresponding line bundles over \mathcal{F}_λ by x_1, \dots, x_n , respectively. Then

$$(8.6) \quad H_\lambda = \mathbb{C}[z_1, \dots, z_n]^{S_n} \otimes \mathbb{C}[x_1, \dots, x_n] \otimes \mathbb{C}[h] \left/ \left\langle \prod_{i=1}^n (u - x_i) = \prod_{i=1}^n (u - z_i) \right\rangle \right.,$$

see (5.1). The assignment $f(z_1, \dots, z_n; x_1, \dots, x_n; h) \mapsto f(x_1, \dots, x_n; x_1, \dots, x_n; h)$ defines an isomorphism H_λ with $\mathbb{C}[x_1, \dots, x_n] \otimes \mathbb{C}[h]$.

Consider the following two \mathfrak{H}_n -actions τ^\pm on $\mathbb{C}[x_1, \dots, x_n] \otimes \mathbb{C}[h] \simeq H_\lambda$. For any $i = 1, \dots, N$, $\tau^\pm(y_i)$ is the multiplication by x_i , and $\tau^\pm(c)$ is the multiplication by $\pm h$. The elements s_1, \dots, s_{n-1} act by the rule

$$\begin{aligned} \tau^\pm(s_i) : f(x_1, \dots, x_n, h) &\mapsto \frac{x_i - x_{i+1} \mp h}{x_i - x_{i+1}} f(x_1, \dots, x_{i+1}, x_i, \dots, x_n, h) \pm \\ &\pm \frac{h}{x_i - x_{i+1}} f(x_1, \dots, x_i, x_{i+1}, \dots, x_n, h), \end{aligned}$$

cf. (2.4). The actions of s_1, \dots, s_{n-1} are uniquely determined by relations in \mathfrak{H}_n , see (8.1), and the property $\tau^\pm(s_i) : 1 \mapsto 1$ for all $i = 1, \dots, n-1$.

Theorem 8.5. *For $i = 1, \dots, n$, we have*

$$\rho^\pm(X_i^q) = \tau^\pm(Y_i^q) = x_i \pm h \sum_{j=1}^{i-1} \frac{q_i}{q_i - q_j} \tau^\pm(s_{i,j}) \pm h \sum_{j=i+1}^n \frac{q_j}{q_i - q_j} \tau^\pm(s_{i,j}).$$

Proof. Since $\rho^\pm(X) = (\nu_\lambda^\pm)^{-1} \phi^\pm(X) \nu_\lambda^\pm$ for any $X \in \tilde{Y}(\mathfrak{gl}_N)$, the statement follows from formula (8.4), Lemma 8.4, and the definition of τ^\pm . \square

In particular, by Theorem 8.5 and formula (3.17), for any $i = 1, \dots, n$, we get

$$(8.7) \quad \rho^-(X_{\lambda,i}^{q-}) = x_i - h \sum_{j=1}^{i-1} \frac{q_i}{q_i - q_j} (\tau^-(s_{i,j}) - 1) - h \sum_{j=i+1}^n \frac{q_j}{q_i - q_j} (\tau^-(s_{i,j}) - 1).$$

8.3. Relations in \mathcal{H}_λ^q and trigonometric Calogero-Moser model. For $N = n$ and $\lambda = (1, \dots, 1)$, recall the algebra \mathcal{H}_λ^q defined in (6.2),

$$\mathcal{H}_\lambda^q = \mathbb{C}[z_1, \dots, z_n]^{S_n} \otimes \mathbb{C}[x_1, \dots, x_n] \otimes \mathbb{C}[h] \left/ \left\langle W^q(u) = \prod_{1 \leq i < j \leq n} (q_i - q_j) \prod_{a=1}^n (u - z_a) \right\rangle \right.$$

where x_1, \dots, x_n denote the Chern roots $\gamma_{1,1}, \dots, \gamma_{n,1}$ and

$$W^q(u) = \det \left(q_i^{n-j} (u - x_i + h(i-j)) \right)_{i,j=1}^n.$$

Lemma 8.6. *The isomorphism $\beta_\lambda^- : H_\lambda \rightarrow \mathcal{H}_\lambda^q$ is such that $\beta_\lambda^-(x_i) = x_i$, $i = 1, \dots, n$. \square*

Proposition 7.5 for the case $N = n$, $\lambda = (1, \dots, 1)$ reads as follows.

Proposition 8.7. *The assignment $\chi^\bullet : x_i \mapsto x_i^\bullet$, $i = 1, \dots, n$, extends uniquely to an isomorphism $\mathcal{H}_\lambda^q \rightarrow \tilde{\mathcal{B}}_\lambda^{q-}$ of $\mathbb{C}[z_1, \dots, z_n]^{S_n} \otimes \mathbb{C}[h]$ -algebras. The maps χ_λ^\bullet and β_λ^- identify the $\tilde{\mathcal{B}}_\lambda^{q-}$ -module H_λ with the regular representation of the algebra \mathcal{H}_λ^q . \square*

Define the $n \times n$ matrix C with entries

$$C_{i,i} = x_i - h \sum_{j=1}^{i-1} \frac{q_i}{q_i - q_j} - h \sum_{j=i+1}^n \frac{q_j}{q_i - q_j}, \quad i = 1, \dots, n,$$

$$C_{i,j} = \frac{h q_i}{q_i - q_j}, \quad i, j = 1, \dots, n, \quad i \neq j.$$

Notice that

$$C_{i,i} = \rho_\lambda^{q-}(X_i^q), \quad i = 1, \dots, n,$$

where X_1^q, \dots, X_n^q are the dynamical Hamiltonians (3.15), see (3.17) and Lemma 6.10.

Lemma 8.8. $W^q(u) = \det(u - C) \prod_{1 \leq i < j \leq n} (q_i - q_j).$

Proof. The verification is an exercise in linear algebra, see [MTV6, Section 6.1]. \square

Recall the function $\widehat{W}^q(u, x)$, see (6.3). Define the diagonal matrix $Q = \text{diag}(q_1, \dots, q_n)$.

Lemma 8.9. $\widehat{W}^q(u) = \det((u - C)(x - Q) - hQ) \prod_{1 \leq i < j \leq n} (q_i - q_j).$

Proof. The statement follows from Lemma 8.8 for $(n+1) \times (n+1)$ matrices by taking an appropriate limit, see similar calculation in [MTV7, Lemma 4.1] for the ordinary Wronskian. \square

Corollary 8.10. *For $N = n$, $\lambda = (1, \dots, 1)$, we have*

$$\mathcal{H}_\lambda^q = \mathbb{C}[z_1, \dots, z_n]^{S_n} \otimes \mathbb{C}[x_1, \dots, x_n] \otimes \mathbb{C}[h] \left/ \left\langle \det((u-C)(x-Q) - hQ) = \prod_{a=1}^n (u - z_a) \right\rangle \right.$$

Cf. this statement with part (4) of Theorem 3.2 in [BMO].

The coefficients of the determinant $\det(u-C)$ in Lemma 8.8 form a complete set of integrals in involution of the trigonometric Calogero-Moser model, see for example [E]. There q_1, \dots, q_n are exponentials of coordinates, and the diagonal entries $C_{1,1}, \dots, C_{n,n}$ are momenta. The Hamiltonian of the trigonometric Calogero-Moser model equals

$$\frac{1}{2} \operatorname{tr} C^2 = \frac{1}{2} \sum_{i=1}^n C_{i,i}^2 + \frac{h^2}{2} \sum_{1 \leq i < j \leq n} \frac{q_i q_j}{(q_i - q_j)^2}.$$

The matrices Q and C satisfy the relation $\operatorname{rank}(CQ - QC - hQ) = 1$. Thus if x_1, \dots, x_n, h are numbers, the matrices Q and C/h define a point of the trigonometric Calogero-Moser space. Notice that the matrices Q and $\tilde{C} = Q^{-1}C/h$ satisfy the relation $\operatorname{rank}(\tilde{C}Q - Q\tilde{C} - 1) = 1$. So the correspondence $(Q, C/h) \mapsto (Q, \tilde{C})$ describes an embedding of the trigonometric Calogero-Moser space into the rational Calogero-Moser space as a submanifold of points with invertible matrix Q .

8.4. Bethe algebra and quantum equivariant cohomology. In [MO], the quantum multiplication on $H_{T^n \times \mathbb{C}^*}^*(T^*\mathcal{F}_\lambda)$ is described. Previously, in [BMO], A. Braverman, D. Maulik and A. Okounkov considered the case $N = n$, $\lambda = (1, \dots, 1)$, of the equivariant cohomology $H_{SL_n \times \mathbb{C}^*}^*(T^*\mathcal{F}_\lambda)$ of the full flag variety and described the associated quantum connection on the trivial bundle with fiber $H_{SL_n \times \mathbb{C}^*}^*(T^*\mathcal{F}_\lambda; \mathbb{C})$ over the space $(\mathbb{C}^*)^{n-1}$, which is the maximal torus of the group SL_n .

Let $N = n$ and $\lambda = (1, \dots, 1)$. Consider the quotient algebras

$$\begin{aligned} \bar{H}_\lambda &= H_\lambda / \langle z_1 + \dots + z_n = 0 \rangle, \\ \bar{\mathcal{B}}^q(\tfrac{1}{D}\mathcal{V}_\lambda^-) &= \tilde{\mathcal{B}}^q(\tfrac{1}{D}\mathcal{V}_\lambda^-) / \langle z_1 + \dots + z_n = 0 \rangle, \quad \bar{\mathcal{B}}_\lambda^{q^-} = \tilde{\mathcal{B}}_\lambda^{q^-} / \langle z_1 + \dots + z_n = 0 \rangle. \end{aligned}$$

We identify the notation of [BMO] with our notation as follows: $H_{SL_n \times \mathbb{C}^*}^*(T^*\mathcal{F}_\lambda; \mathbb{C}) = \bar{H}_\lambda$, $\kappa = -1$, $t = -h$, $q^{\alpha_i^\vee} = q_{i+1}/q_i$, $i = 1, \dots, n-1$, where $\alpha_1, \dots, \alpha_{n-1}$ are simple roots of the Lie algebra \mathfrak{sl}_n , and all functions of q_1, \dots, q_n are of homogeneous degree zero. Then the quantum connection ∇^{BMO} of [BMO] takes the form

$$(8.8) \quad \nabla_{\alpha_i}^{\text{BMO}} = q_i \frac{\partial}{\partial q_i} - q_{i+1} \frac{\partial}{\partial q_{i+1}} - (x_i - x_{i+1}) \bullet, \quad i = 1, \dots, n,$$

see (7.2), (8.7).

Lemma 8.11. *The algebra $\bar{\mathcal{B}}_{\lambda}^{q-}$ is generated by the elements $\rho_{\lambda}^{-}(X_{\lambda,i}^{q-} - X_{\lambda,i+1}^{q-})$, $i = 1, \dots, n-1$, and $\mathbb{C}[z_1, \dots, z_n]^{S_n} \otimes \mathbb{C}[h]$.*

Proof. By Theorem 8.1, the algebra $\bar{\mathcal{B}}^q(\frac{1}{D}\mathcal{V}_{\lambda}^{-})$ is generated by the elements $\phi^{-}(X_{\lambda,i}^{q-})$, $i = 1, \dots, n$, and $\mathbb{C}[z_1, \dots, z_n]^{S_n} \otimes \mathbb{C}[h]$. The lemma follows from the identity $\phi^{-}(X_{\lambda,1}^{q-} + \dots + X_{\lambda,n}^{q-}) = z_1 + \dots + z_n$, see formula (3.17). \square

The algebra of quantum multiplication on \bar{H}_{λ} determined in [BMO] is generated by the elements $(x_i - x_{i+1}) \bullet$, $i = 1, \dots, n-1$, see [BMO]. Thus by Corollary 7.3 and Lemma 8.11, we have the following result verifying Theorem 7.10 for $N = n$ and $\lambda = (1, \dots, 1)$.

Theorem 8.12. *For $N = n$, $\lambda = (1, \dots, 1)$, the quantum multiplication on \bar{H}_{λ} determined in [BMO] coincides with the multiplication \bullet on \bar{H}_{λ} introduced in (7.1).*

Consider the quotient algebra $\bar{\mathcal{H}}_{\lambda}^q = \mathcal{H}_{\lambda}^q / \langle z_1 + \dots + z_n = 0 \rangle$. Since $x_1 + \dots + x_n = z_1 + \dots + z_n$, the algebra $\bar{\mathcal{H}}_{\lambda}^q$ is generated by the elements $x_i - x_{i+1}$, $i = 1, \dots, n-1$. By Proposition 8.7 we have the following result.

Corollary 8.13. *For $N = n$, $\lambda = (1, \dots, 1)$, the space H_{λ} as the module over the algebra of quantum multiplication on H_{λ} defined in [BMO] is isomorphic to the regular representation of the algebra $\bar{\mathcal{H}}_{\lambda}^q$ by the isomorphisms $x_i - x_{i+1} \mapsto x_i - x_{i+1}$, $(x_i - x_{i+1}) \bullet \mapsto x_i - x_{i+1}$, $i = 1, \dots, n-1$.*

It is known that the flat sections of the trigonometric dynamical connection are given by multidimensional hypergeometric integrals, see [MV], cf. [TV2]. These hypergeometric integrals provide flat sections of the quantum connection defined by (3.20), see [TV3]. This presentation of flat sections of the quantum connection as hypergeometric integrals is in the spirit of the mirror symmetry, see Candelas et al. [COGP], Givental [G1, G2], and [BCK, BCKS, GKLO, I, JK].

Example. Let $N = n = 2$ and $\lambda = (1, 1)$. Set $x = x_1 - x_2$, $z = z_1 - z_2$, $q = q_2/q_1$. Then

$$\bar{H}_{\lambda} = \mathbb{C}[x, z^2, h] / \langle x^2 = z^2 \rangle, \quad \bar{\mathcal{H}}_{\lambda}^q = \mathbb{C}[x, z^2, h] / \left\langle x^2 - \frac{4hq}{1-q}(x+h) = z^2 \right\rangle,$$

and

$$x \bullet = x - \frac{2hq}{1-q}(\tau^{-}(s) - 1)$$

is the operator of quantum multiplication by x acting on \bar{H}_{λ} . Here the operator $\tau^{-}(s)$ is

$$\tau^{-}(s) : f(x, z^2, h) \mapsto \frac{x+h}{x} f(-x, z^2, h) - \frac{h}{x} f(x, z^2, h).$$

The isomorphism of the algebra of quantum multiplication on \bar{H}_{λ} and $\bar{\mathcal{H}}_{\lambda}^q$ is $x \bullet \mapsto x$, that is, the operator $x \bullet$ acting on \bar{H}_{λ} satisfies the equation

$$(x \bullet)^2 - \frac{4hq}{1-q}(h + x \bullet) = z^2.$$

The quantum differential equation is

$$(8.9) \quad -2\kappa q \frac{\partial}{\partial q} I = x \bullet I.$$

The q -hypergeometric solutions of this equation are given by the formula

$$I = q^{z/2\kappa} (x+h) \int_C q^{u/\kappa} \Gamma\left(\frac{u-h}{\kappa}\right) \Gamma\left(\frac{u+z-h}{\kappa}\right) \Gamma\left(-\frac{u}{\kappa}\right) \Gamma\left(-\frac{u+z}{\kappa}\right) (2u+z-x) du,$$

where the integration contour C is a deformation of the imaginary line separating the poles of $\Gamma((u-h)/\kappa) \Gamma((u+z-h)/\kappa)$ from those of $\Gamma(-u/\kappa) \Gamma(-(u+z)/\kappa)$. Linearly independent solutions can be obtained by choosing various branches of $q^{u/\kappa}$.

Alternatively, a basis of solutions can be obtained by taking the sum of residues of the integrand at $u \in \kappa\mathbb{Z}_{\geq 0}$ or at $u \in -z + \kappa\mathbb{Z}_{\geq 0}$. The sum of residues at $u \in \kappa\mathbb{Z}_{\geq 0}$ gives

$$\begin{aligned} J_1 &= (x+h) \Gamma\left(-\frac{h}{\kappa}\right) \Gamma\left(-\frac{z}{\kappa}\right) \Gamma\left(\frac{z-h}{\kappa}\right) \times \\ &\quad \times \sum_{d=0}^{\infty} q^{d+z/2\kappa} \frac{z-x+2\kappa d}{\kappa^{d-1} d!} \prod_{i=0}^{d-1} \frac{(h-\kappa i)(h-z-\kappa i)}{z+\kappa(i+1)}, \end{aligned}$$

and the sum of residues at $u \in -z + \kappa\mathbb{Z}_{\geq 0}$ gives

$$\begin{aligned} J_2 &= (x+h) \Gamma\left(-\frac{h}{\kappa}\right) \Gamma\left(\frac{z}{\kappa}\right) \Gamma\left(-\frac{z+h}{\kappa}\right) \times \\ &\quad \times \sum_{d=0}^{\infty} q^{d-z/2\kappa} \frac{(2\kappa d - z - x)}{\kappa^{d-1} d!} \prod_{i=0}^{d-1} \frac{(h-\kappa i)(h+z-\kappa i)}{\kappa(i+1) - z}. \end{aligned}$$

The hypergeometric solutions of equation (8.9) are given by the formula

$$I = q^{z/2\kappa} (1-q)^{2h/\kappa} \int_C u^{(h-z)/\kappa} (u-1)^{-h/\kappa} (u-q)^{-h/\kappa} \left(\frac{x-z+2h}{u-1} + \frac{x+z}{u-q} \right) du,$$

where C is an appropriate integration contour. We plan to discuss q -hypergeometric and hypergeometric formulae for flat sections of the quantum connection in a separate paper.

The dynamical connection commutes with the difference qKZ connection as explained in Section 4.5. For $N = n$, $\lambda = (1, \dots, 1)$, the dynamical connection is identified with the quantum connection by Theorem 8.12. Then the qKZ connection of Section 4.5 induce on H_λ a difference connection that commutes with the quantum differential connection in the sense of Section 4.5. This discrete structure was discussed in [BMO] under the name of shift operators. We will discuss the qKZ discrete connection in this situation in more details in another paper.

The isomorphism of the Bethe algebra and quantum multiplication establishes a correspondence between the eigenvectors of the operators of the Bethe algebra and idempotents

of the quantum algebra. Hence we may find the idempotents of the quantum algebra by the XXX Bethe ansatz.

Example. Let $N = n = 2$, $\lambda = (1, 1)$. If u_1, u_2 are solutions of the Bethe ansatz equation

$$(u - z)(u + z) = q(u - z + 2h)(u + z + 2h),$$

then the elements

$$w_1 = \frac{x - u_2}{u_1 - u_2}, \quad w_2 = \frac{x - u_1}{u_2 - u_1}$$

of \bar{H}_λ satisfy the equations $w_i \bullet w_j = \delta_{i,j} w_i$.

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